LOOPBACK FREQUENCY MODULATION USING A TIME-VARYING DELAY LINE

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ABSTRACT

This work examines the use of the time-varying delay line (TVDL) to implement loopback frequency modulation (LBFM), an oscillator that loops back to modulate its own frequency. Digital delay lines are used regularly in sound synthesis/processing to model the pure delay associated with one-dimensional acoustic propagation. When the delay is made time varying, the TVDL time warps the input according to a delay function, altering the input's instantaneous frequency and phase. As a result, the TVDL is well suited for delay-based effects and, in particular, those involving frequency/phase modulation for which the TVDL delay function is oscillatory and thus bounded by a maximum and minimum delay. Sustaining a constant change in sounding frequency however, corresponds to a delay function having a term that is linear in time, making it limited only by the length of the input signal. While TVDLs may still be used when there is a pitch shift, limiting the delay by simple wrapping of the delay function and/or cross fading between multiple TVDLs may not be adequate to avoid audible artifacts. In LBFM, the resulting phase has both linear and oscillating terms and the resulting signal undergoes a sustained shift in the fundamental frequency that makes a TVDL implementation more challenging. An alternate closed-form representation of the LBFM oscillator, however, provides the information necessary for accurately wrapping the TVDL delay function and ensuring it is suitably bounded so that the produced sound is free of phase distortion and audible artifacts.

1. INTRODUCTION

Frequency modulation (FM) [1], a near ubiquitous synthesis technique in the field of Computer Music, is often presented as the phase modulation (PM) of a sinusoid, with the sinusoid's instantaneous phase being the integral of the instantaneous frequency over time. The PM formalation is preferred for its improved numerical accuracy and an intuitive control parameter, the *index of modulation*, giving an approximation to the number of audible sidebands the synthesis technique produces in the spectrum.

Loopback frequency modulation (LBFM), a term originally coined in [2], refers to an oscillator that loops back to modulate its own instantaneous frequency, introducing a nonlinearity to the system governing its motion. The term was introduced to distinguish it from feedback FM [3], a technique whereby the signal feeds back to modulate its own phase, with the name likely stemming from directly applying feedback to an "FM oscillator" that is actually implemented as PM (modulating the frequency instead would require an additional step of integration to obtain the phase of the sinusoid). More recently, the term feedback FM has been used in [4] to mean the same as LBFM here (modulating frequency rather than phase), adopting the disambiguating term feedback PM while illustrating the waveforms and spectra produced by each and keeping a consistent naming scheme with the closely related feedback AM (FBAM) [5, 6].

Feedback systems have long been a subject of great interest in many musical contexts: from a guitarist playing near a speaker with a high amplifier gain, to experimental uses of acoustic feedback as source material in works by composers like Cage, Tudor, Lucier and Reich, to Computer Music research where feedback is used to create recursive structures such as those involving delay lines, applying them to the synthesis of periodic tones [7] or, in larger networks, creating audio effects like improved colorless reverberation [8]. LBFM is a continuation of ongoing research exploring feedback, with the goal of efficiently introducing the audible/musical effects associated with nonlinearities (e.g. time-varying amplitude and spectra, frequency sweeps) into a system of one or more coupled oscillators. While nonlinear should not necessarily be conflated with feedback, it is well-known that nonlinearities may be introduced by placing a system in a feedback configuration (as, for example, when the displacement of a simple harmonic oscillator is made to modify its natural resonant frequency, a value that is squared in its governing equation of motion [9]).

Summarized in Section 2, LBFM was initially presented in [2] as its own abstract synthesis method (starting from sinusoids) with three implementations, along with mappings between their corresponding parameters. The first was presented as FM, expressed as a sample-by-sample complex rotation of its current state and showing clearly how the carrier oscillator is "looped back" with a coefficient to serve as the modulator of its own instantaneous frequency. The second implementation was presented as PM, where frequency is integrated over time to obtain phase, with the modulating part—the oscillator itself having yet no known analytic expression—integrated numerically. While both FM and PM implementations of LBFM are shown to be equivalent (the sample-by-sample rotation in FM and the numerical integration of frequency by continuous summation in PM yield the same final expression), a third implementation presents a closed-form solution with improved accuracy of the delay-free loop for which both FM and PM implementations (and the work described in [4]) necessarily introduce a unit sample delay. Further, the closed-form solution was shown to have two additional benefits: 1) providing an analytic representation of the phase and 2) being a function

* Thanks to the predecessors for the templates

of, and thus revealing, the more musically meaningful parameter of sounding frequency which is sometimes difficult to predict in nonlinear systems. While the former benefit was shown to improve numerical accuracy, the latter greatly facilitated a musical exploration of LBFM in extended modal synthesis of percussion instruments, incorporating parametric time variation characteristic of nonlinearities (e.g. change in spectral bandwidth and pitch glides like those heard in gongs) without issues of stability and computational cost [10]. It has since become evident that these benefits (analytic phase and fundamental sounding frequency) also make a fourth implementation of LBFM possible, using a time-varying delay line (TVDL), thus allowing for arbitrary audio input and further expanding the audio synthesis/processing contexts in which the technique may be explored.

As discussed in Section 3, a TVDL time warps an input signal based on a time-varying delay function obtained by integrating a specified relative frequency shift over time. Since the relative frequency shift is a function of the instantaneous frequency, integration yields an expression for the delay function that is comprised of instantaneous phase, which may be obtained from the analytic phase of the closed-form solution mentioned above (and described in Section 2). Moreover, since LBFM involves a sustained transposition of frequency, the relative frequency shift is comprised of a scalar value that when integrated yields a delay function that increases linearly with the time sample. This presents a difficulty when implementing with a circular delay line having a delay function that must be limited by a maximum and minimum value. While solutions have been suggested for sustaining a pitch shift using TVDLs (e.g. crossfading between more than one delay line [11]), the effectiveness at attenuating audible artifacts is largely dependent on context and the nature of the sound being processed. Instead, a solution is presented in Section 3 for warping the delay function so that it moves between a minimum and maximum delay value ever N samples, where the value of N is determined based on the known phase and frequency of the closed-form solution for LBFM.

2. LOOPBACK FREQUENCY MODULATION (LBFM)

It is well known that in frequency modulation (FM) synthesis, a sinusoid has a carrier frequency that is modulated by another real sinusoid, resulting in a signal with a time-varying instantaneous frequency that may be represented by

$$\omega_i(t) = \omega_c - d\cos(\omega_m t) \quad \text{rad/s}, \tag{1}$$

where d is the modulator's peak frequency deviation from carrier frequency ω_c and ω_m is the frequency of modulation in rad/s. The instantaneous phase corresponding to (1) is given by integrating (1) with respect to time to yield:

$$\theta_i(t) = \int_0^t \omega_i(t)\, dt = \omega_c t - I\sin(\omega_m t) + \phi_c, \tag{2}$$

where the amplitude of the sinusoidal term

$$I = \frac{d}{\omega_m}, \tag{3}$$

is known as the index of modulation because of its influence on the sidebands that are produced at frequencies $\omega_c \pm k\omega_m$ in the resulting spectrum. If ω_m is in the audio range, changing I can result in a significant perceptual change in both tone quality (timbre) and sounding frequency.

In addition to making use of the intuitive parameter I, its value being closely related to the number of audible sidebands, implementing FM as phase modulation (PM) by using (2) is known for having improved accuracy—with an analytic rather than numeric integration of frequency being independent of a sampling rate f_s. If the phase (2) is known (and modulation is not on the frequency of an existing and unknown input signal), it may simply be made the argument of a complex sinusoid:

$$z_c(t) = e^{j\theta_i(t)}, \tag{4}$$

with either the real or imaginary part serving as the produce sound. Similarly, the phase $\theta_i(t)$ in (2) may also be expressed using either the real or imaginary part of a complex modulating sinusoid, expressed here using the latter:

$$\theta_i(t) = \omega_c t + I\Im\{z_m(t)\} + \phi_c \quad \text{for} \quad z_m(t) = e^{j\omega_m t}, \tag{5}$$

a representation that will be useful in later discussion.

2.1. Discrete-Time Implementation of FM

If the analytic expression for the phase in (2) is unknown, the FM oscillator may be implemented in discrete time as a sample-by-sample rotation, a complex multiply of the oscillator's current state at time sample n to produce the oscillator's value at the next time step. That is, beginning with (4) and (5) sampled by the substitution

$$t \longrightarrow nT \quad \text{for } n = 0, 1, \ldots, N-1 \tag{6}$$

for sampling period $T = 1/f_s$ and signal length N, the PM oscillator at time sample $n+1$ (and $\phi_c = 0$ for brevity) is

$$\begin{aligned} z_c(n+1) &= e^{j(\omega_c(n+1)T + I\Im\{z_m(n+1)\})} \\ &= e^{j(\omega_c T + I\Im\{z_m(n+1)\} + \omega_c nT)}. \end{aligned} \tag{7}$$

Adding and subtracting $\Im\{z_m(n)\}$ to the angle of (7) yields the sample-by-sample rotation implementation of (4) given by

$$\begin{aligned} z_c(n+1) &= e^{j(\omega_c T + \Im\{z_m(n+1)\} - \Im\{z_m(n)\})} z_c(n) \\ &= e^{j\omega_i(n+1)T} z_c(n), \end{aligned} \tag{8}$$

where, notably, the instantaneous frequency

$$\omega_i(n+1) = \omega_c + \frac{\Im\{z_m(n+1)\} - \Im\{z_m(n)\}}{T} \tag{9}$$

is equivalent to a first-order finite difference approximation of the time-derivative of the phase (5) which, when multiplied by the sampling period T, yields the angle of rotation.

2.2. The Loopback FM (LBFM) Oscillator

In loopback FM (LBFM), the oscillator $z_b(t)$ loops or feeds back with a scalar coefficient B (though time-varying cases are also possible [2]) such that its real part modulates its carrier frequency ω_c, resulting in the instantaneous frequency

$$\omega_b(t) = \omega_c + B\omega_c\Re\{z_b(t)\}. \tag{10}$$

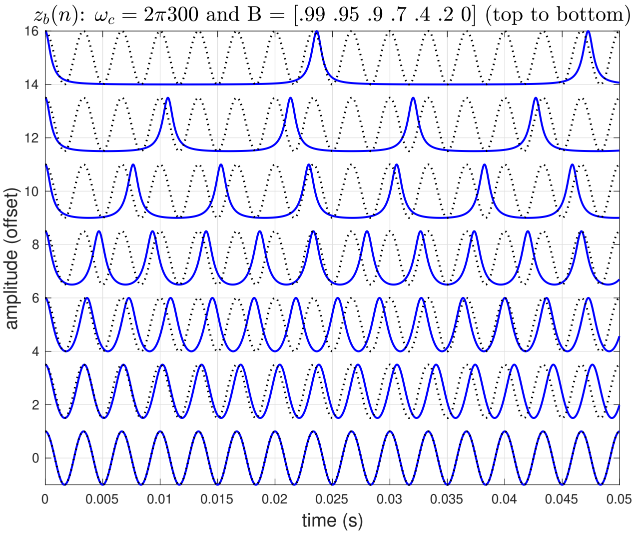

Figure 1: Loopback FM oscillator $z_b(n)$ is shown for several values of feedback coefficient B. As the values of B approach 0, the waveform's shape and fundamental frequency converge to that of a pure cosine wave at frequency ω_c (bottom and dotted curves).

Like in (8), the discrete-time LBFM oscillator may be implemented via a sample-by-sample rotation given by

$$z_b(n) = e^{j\,(\omega_c + B\omega_c \Re\{z_b(n-1)\})\,T} z_b(n-1), \tag{11}$$

with a unit-sample-delay introduced in the angle of rotation for implementation (though a delay-free solution is given below). The modulation amplitude $B\omega_c$ in (11) is the peak frequency deviation from ω_c with the loopback coefficient B being similar to the index of modulation in (3) save its constraint, $-1 < B < 1$, for oscillation to occur. As shown in Figure 1 for a static carrier frequency of 300 Hz, changing the loopback coefficient B has the effect of changing the period and thus the fundamental frequency of the waveform, a relationship that will be derived below as

$$\omega_0 = \pm\omega_c\sqrt{1 - B^2}. \tag{12}$$

Equation (12) shows that smaller values of B yield a fundamental frequency that is closer to the carrier frequency ω_c, a fact that may also be observed in Figure 1 along with the corresponding convergence in shape to a cosine waveform at frequency ω_c (dotted curves). Like modulation index I, coefficient B alters the period and shape of the waveform and thus its frequency content but, unlike I, it requires a much smaller range to do so. Further, index I is not as strictly limited as B and is bounded only by the fact that the sidebands it introduces will cause aliasing if not kept below the Nyquist limit.

2.3. LBFM Implemented as Phase Modulation

Integrating (10) with respect to time yields instantaneous phase

$$\theta_b(t) = \int_0^t \omega_b(t)\,dt = \omega_c t + B\omega_c \Re\left\{\int_0^t z_b(t)\,dt\right\}, \tag{13}$$

an integration that is not possible analytically without a known expression for $z_b(t)$. A numerical solution via a cumulative sum yields a discrete-time approximation to (13) that is given by

$$\theta_b(n) = \omega_c nT + B\omega_c T \Re\left\{\sum_{k=0}^{n-1} z_b(k)\right\}, \tag{14}$$

which may be made the argument of a sinusoid as in (4) to implement LBFM via phase modulation (PM). This implementation would not, however, improve upon the accuracy of (11) and, in fact, may be shown to be equivalent. That is, starting with (14) and separating the last element of the sum,

$$\theta_b(n) = \omega_c nT + B\omega_c T \Re\left\{z_b(n-1) + \sum_{k=0}^{n-2} z_b(k)\right\}, \tag{15}$$

then subtracting and adding $\omega_c(n-1)T$ to obtain the expression

$$\theta_b(n) = \omega_c T + B\omega_c T \Re\{z_b(n-1)\} + \theta_b(n-1), \tag{16}$$

then making it the argument of a complex exponential sinusoid

$$\begin{aligned} e^{j\theta_b(n)} &= e^{j\,(\omega_c T + B\omega_c T \Re\{z_b(n-1)\})} e^{j\theta_b(n-1)} \\ &= z_b(n), \end{aligned} \tag{17}$$

yields an expression that is equivalent to the complex multiply (sample-by-sample rotation) implementation of FM given in (11). This result shows, as in the case of regular FM for (7)–(8), that a PM implementation in which frequency is numerically integrated to obtain phase, is equivalent and thus no more advantageous than the FM implementation. A more accurate representation of (11) is expected, therefore, with an analytic integration of (10) to obtain the oscillator's instantaneous phase. Such a solution is provided in [2] whereby an alternate closed-form expression for (11) not only allows for its analytic integration, but also improves accuracy of the LBFM oscillator by avoiding the unit-sample delay in an otherwise delay-free loop and, furthermore, is a function of the more meaningful parameter of the sounding frequency ω_0 instead of a carrier frequency for which the sounding frequency must be determined. In addition to being more musically useful, this latter advantage of providing knowledge of the sounding frequency allows this representation of the LBFM oscillator to inform the solution for its implementation as a TVDL as discussed in Section 3.

2.4. The Alternate Closed-Form Solution to LBFM

As shown in [2], the oscillator $z_b(n)$ has a period of M samples or M/f_s seconds and a corresponding fundamental frequency of $\omega_0 = 2\pi f_s/M$ rad/s, the real part producing a waveform with the same shape as the real part given by the closed-form expression, given in continuous and discrete-time by

$$z_0(t) = \frac{b_0 + e^{j\omega_0 t}}{1 + b_0 e^{j\omega_0 t}} \quad \text{and} \quad z_0(n) = \frac{b_0 + e^{j\omega_0 nT}}{1 + b_0 e^{j\omega_0 nT}}, \tag{18}$$

respectively. The instantaneous phase of (18), familiar due to the similar construction of (18) to the transfer function of an all-pass filter, is given in continuous time by

$$\angle z_0(t) = \omega_0 t - 2\phi(t), \tag{19}$$

where

$$\phi(t) = \tan^{-1}\left(\frac{b_0 \sin(\omega_0 t)}{1 + b_0 \cos(\omega_0 t)}\right). \tag{20}$$

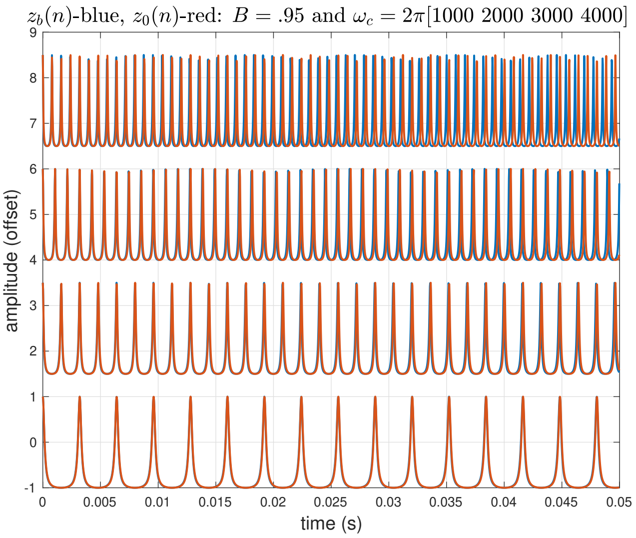

Figure 2: Loopback FM oscillator $z_b(n)$ is shown with its alternate closed-form representation $z_0(n)$ for several values of carrier frequencies ω_c. Notice that while there is good agreement at lower frequencies (bottom), there is noticeable drift between them a higher frequencies (top).

It was shown in [12] that while the waveforms of $z_0(n)$ and $z_b(n)$ diverge for increasing values of ω_c and B (see Figure 2 for increasing ω_c), this discrepancy is greatly reduced by increasing the sampling rate in $z_b(n)$, suggesting that $z_0(n)$ is the more accurate representation because of its independence on the sampling rate.

The alternate expression for the LBFM oscillator allows the integral in (13) to be solved analytically, a process that also leads to a mapping between parameters B and ω_c of $z_b(n)$ and parameters b_0 and ω_0 of $z_0(n)$. The integral of continuous-time $z_0(t)$ w.r.t. time in (18) is given by

$$\begin{aligned}\int_0^t z_0(t)\,dt &= b_0 t + \frac{1-b_0^2}{j\omega_0 b_0}\log\left(1+b_0 e^{j\omega_0 t}\right)\\ &= b_0 t + \frac{1-b_0^2}{j\omega_0 b_0}\log\left(A(t)e^{j\phi(t)}\right)\\ &= b_0 t + \frac{1-b_0^2}{j\omega_0 b_0}\left(\log(A(t)) + j\phi(t)\right), \quad (21)\end{aligned}$$

with the real part reducing to

$$\Re\left\{\int_0^t z_0(t)\,dt\right\} = b_0 t + \frac{1-b_0^2}{\omega_0 b_0}\phi(t), \tag{22}$$

where $\phi(t)$ was given by (20). Substituting (22) into the real part of the integral term in (13 yields an expression for the instantaneous phase of $z_b(t)$ given by

$$\theta_b(t) = \omega_c t(1+Bb_0) + B\omega_c\frac{1-b_0^2}{\omega_0 b_0}\phi(t) \tag{23}$$

which, when made equal to the phase of closed form solution given in (19) yields the expression

$$\omega_c t(1+Bb_0) + B\omega_c\frac{1-b_0^2}{\omega_0 b_0}\phi(t) = \omega_0 t - 2\phi(t). \tag{24}$$

Making linear and oscillating terms on either side of (24) equal, yields two expressions for ω_0 given by

$$\omega_0 = \omega_c(1+Bb_0) \quad \text{and} \quad \omega_0 = \frac{\omega_c B(1-b_0^2)}{-2b_0}, \tag{25}$$

which when made equal, result is a quadratic equation with solution given by

$$b_0 = \frac{\pm\sqrt{1-B^2}-1}{B}, \quad \text{for } B \neq 0. \tag{26}$$

Finally, substituting (26) into either expression in (25) yields the expression for ω_0 as a function of LBFM parameters B and ω_c that was given previously in (12):

$$\omega_0 = \omega_c\left(1 + B\frac{\pm\sqrt{1-B^2}-1}{B}\right) = \pm\omega_c\sqrt{1-B^2}. \tag{27}$$

3. TIME-VARYING DELAY LINE (TVDL) AND LBFM

It is well known that a circular delay line is a computationally efficient way to apply a pure delay to an input signal in the time domain. To implement a circular delay line, a buffer is allocated having a size (sample length) corresponding to the maximum desired delay D_M. To implement an actual delay D, consecutive audio samples written to the buffer at incremental positions are read at a position trailing by D samples. Both writing and reading positions may wrap around either end of the buffer (thus making it circular), ensuring the actual delay D is limited by

$$d_0 \leq D(n) \leq D_M \tag{28}$$

allowing for possible offset d_0 and variation with time sample n. When a delay line is made time varying at an audio rate, the delay is no longer necessarily "pure" and the input signal is likely to be warped in time (and its wave shape altered). For example, if the input into a TVDL with delay function $D(n)$ is the complex sinusoid

$$x(n) = e^{j\omega_c nT} \tag{29}$$

with frequency ω_c, then the delay line output

$$y(n) = x(n - D(n)) = e^{j\theta_c(n)} \tag{30}$$

has instantaneous phase

$$\theta_c(n) = \omega_c(n - D(n))T, \tag{31}$$

showing instances of n in the input being substituted by warping function $n - D(n)$ in the output. Taking the derivative of (31) w.r.t. (discrete) time nT yields the output's instantaneous frequency

$$\omega_i(n) = \frac{d}{dnT}\omega_c(n - D(n))T = \omega_c\left(1 - \frac{d}{dn}D(n)\right). \tag{32}$$

To set $D(n)$ according to some desired output frequency $\omega_i(n)$, the expression (32) is rearranged to isolate the delay's derivative,

$$\frac{d}{dn}D(n) = 1 - \frac{\omega_i(n)}{\omega_c} = 1 - m_t(n), \tag{33}$$

also known as the relative frequency shift for momentary transposition $m_t(n) = \omega_i(n)/\omega_c$ in [13, 11], so that the TVDL's corresponding delay function is obtained by integrating (33) over time sample n:

$$D(n) = \int_0^n \left(1 - \frac{\omega_i}{\omega_c}\right)dn = n - \frac{\theta_c}{\omega_c}f_s. \tag{34}$$

3.1. Phase and Frequency Modulation using a TVDL

If the desired instantaneous frequency is the modulated frequency from (1) expressed in discrete time with the substitution $t \longrightarrow nT$,

$$\omega_i(n) = \omega_c - d\cos(\omega_m nT) \text{ rad/s}, \tag{35}$$

then the momentary transposition is

$$m_t(n) = \frac{\omega_i(n)}{\omega_c} = 1 - \frac{d}{\omega_c}\cos(\omega_m nT). \tag{36}$$

Substituting $d = I\omega_m$ from (3) into (36), the relative frequency shift as given by the relation in (33) becomes

$$\frac{d}{dn}D_m(n) = 1 - m_t(n) = I\frac{\omega_m}{\omega_c}\cos(\omega_m nT)) \tag{37}$$

which is integrated over sample n to yield the delay function

$$D_m(n) = \frac{I}{\omega_c}\sin(\omega_m nT)f_s + \frac{I}{\omega_c}f_s. \tag{38}$$

The DC offset of If_s/ω_c in (38) may be interpreted as a constant of integration that, by being equal to the amplitude of the sinusoidal term in (38), ensures the positive delay function necessary for implementation using a circular delay line. Alternatively, the delay function may be obtained using an equivalent PM approach whereby the desired frequency $\omega_i(n)$ in (35) is integrated over time to obtain instantaneous phase as in (2) but given here in discrete time,

$$\theta_i(n) = \omega_c nT - I\sin(\omega_m nT) + \phi_c, \tag{39}$$

and then substituting it for θ_c in (34) to yield

$$\begin{aligned} D_m(n) &= n - \frac{\theta_i(n)}{\omega_c}f_s \\ &= \frac{I}{\omega_c}\sin(\omega_m nT)f_s - \frac{\phi_c}{\omega_c}f_s, \end{aligned} \tag{40}$$

which reduces to (38) when $\phi_c = -I$. The above steps are validated by showing that a TVDL input of (29) produces output

$$y_m(n) = x(n - D_m(n)) = e^{j\omega_c(n - D_m(n))T} \tag{41}$$

where the resulting instantaneous phase

$$\begin{aligned} \theta_m(n) &= \omega_c\left(n - \omega_c D_m(n)\right)T \\ &= \omega_c nT - \omega_c\left(\frac{I}{\omega_c}\sin(\omega_m nT) - \frac{\phi_c}{\omega_c}\right) \\ &= \omega_c nT - I\sin(\omega_m nT) + \phi_c, \end{aligned} \tag{42}$$

is the discrete-time representation of (2) given in (39).

3.2. TVDL Delay Wrapping with Constant Frequency Shift

Since LBFM involves a sustained frequency shift, its relative frequency is comprised of a scalar value that when integrated, yields a delay function that increases with time sample n, a caveat when implementing with a circular delay line of finite length. Here, a solution is presented for wrapping the delay function so that it satisfies (28). For simplicity and ease of discussion, the solution is presented with a sinusoid, but it is easily applied to LBFM using the closed-form solution which provides both analytic phase and sounding frequency.

If TVDL output sinusoid is a sustained transposition of input sinusoid (29), the desired frequency will be a scalar value ω_i which, when integrated, yields a corresponding phase that is linear in time:

$$\theta_l(n) = \omega_i nT. \tag{43}$$

Substituting (43) into (34) yields the delay function

$$D_l(n) = n - \frac{\omega_i nT}{\omega_c}f_s = n\left(1 - \frac{\omega_i}{\omega_c}\right), \tag{44}$$

which increases linearly over time sample n. Delay function (44) is shown to be accurate by replacing instances of n with warping function $n - D_l(n)$ for input (29), yielding

$$\begin{aligned} y_t(n) &= e^{j\omega_c(n - D_l(n))T} \\ &= e^{j\omega_c\left(n - \left(n - n\frac{\omega_i(n)}{\omega_c}\right)\right)T} = e^{j\theta_l(n)}, \end{aligned} \tag{45}$$

a sinusoid with phase equivalent to (43). However the delay function (44) is unbounded and does not satisfy (28), making it unsuitable for implementation with a circular delay line. Sustaining a constant transposition with a linear delay function requires additional treatment to ensure it is both positive and limited by the length of the (circular) delay line. In [11], a solution is suggested using two delay lines with delay function being arbitrarily wrapped to satisfy (28), crossfading between the two TVDL outputs to attenuate any artifacts caused by discontinuities in the delay function. While this approach may be effective for more incoherent signals (e.g. audio recordings), it was found to have limited success here. Instead, it is proposed that a single delay line be used, but with a delay function that is wrapped more strategically so that the phase of the TVDL output is continuous and no discontinuities and/or artifacts are generated in the produced sound.

Wrapping (44) makes a bounded delay function that satisfies (28) by moving between a minimum and maximum delay value (or vice versa depending on the transposition direction) every N samples, where N is an integer that is obtained by either downward or upward rounding of non-integer $\tilde{N}$, that is

$$N = \{\lfloor\tilde{N}\rfloor, \lceil\tilde{N}\rceil\} \text{ samples}. \tag{46}$$

Non-integer $\tilde{N}$ is found by first considering a wrapping of sample numbers $n = 0, 1, 2, \ldots$ according to

$$\hat{n} = n \bmod \tilde{N}, \text{ for } 0 \leq \hat{n} \leq \tilde{N}, \tag{47}$$

which, when substituted for n in (44) yields a preliminary wrapped delay function

$$\hat{D}_l(n) = \hat{n}\left(1 - \frac{\omega_i}{\omega_c}\right), \tag{48}$$

such that $\hat{D}_l(\tilde{N}) = 0$. It follows that the output of the TVDL with delay function (48) is equal to input (29) every $\tilde{N}$ samples, that is

$$e^{j\omega_c(n - \hat{D}(\tilde{N}))T} = e^{j\omega_c nT}. \tag{49}$$

Finding that value of $\tilde{N}$ for which (49) is satisfied and also for which intersecting output and input signals are simultaneously either rising or falling, meeting the additional signal slope condition

$$\text{sgn}\left(\frac{d}{dn}e^{j\omega_c nT}\right) = \text{sgn}\left(\frac{d}{dn}e^{j\omega_i nT}\right), \tag{50}$$

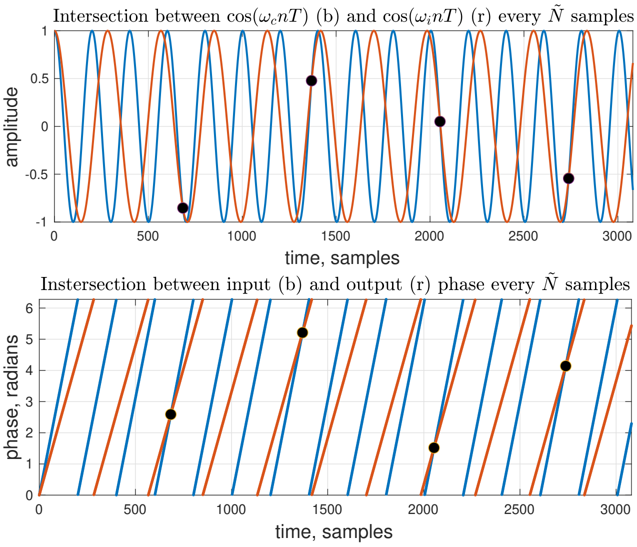

Figure 3: Points of intersection between TVDL input signal $\Re\{e^{j\omega_c nT}\}$ (blue) and output $\Re\{e^{j\omega_i nT}\}$ (red) occurring every N samples. While additional points intersect and satisfy (49), they are disqualified for not also meeting the slope condition in (50). Corresponding points of intersection between the input and output phase (bottom) according to (52) occur only when both conditions are met.

yields a wrapped delay function (48) that avoids discontinuities in the TVDL output and where

$$e^{j\omega_c(n - D_l(n))T} \approx e^{j\omega_c(n - \hat{D}_l(n))T}. \tag{51}$$

Since a sinusoid is periodic with a period of 2π radians, the value for $\tilde{N}$ satisfying two conditions (49) and (50) is also found by satisfying a single phase condition given by

$$\omega_c \tilde{N} T \bmod 2\pi = \omega_i \tilde{N} T \bmod 2\pi, \tag{52}$$

or equivalently

$$\omega_c \tilde{N} T - 2\pi \left\lfloor \frac{\omega_c \tilde{N} T}{2\pi} \right\rfloor = \omega_i \tilde{N} T - 2\pi \left\lfloor \frac{\omega_i \tilde{N} T}{2\pi} \right\rfloor, \tag{53}$$

which may be rearranged and expressed with frequencies in Hz:

$$(f_c - f_i)\tilde{N} T = \lfloor f_c \tilde{N} T \rfloor - \lfloor f_i \tilde{N} T \rfloor, \tag{54}$$

where $f_{c,i} = \omega_{c,i}/(2\pi)$ Hz where (54) is satisfied by

$$\tilde{N} = \left| \frac{f_s}{f_c - f_i} \right|. \tag{55}$$

The points of intersection satisfying conditions (49) and (50) are shown in Figure 3 (top) along with corresponding points of intersection in the phase satisfying (52) (bottom). In a practical implementation, the period of the wrapped signal must be an integer number of samples and so integer N takes on either the floor or ceiling of $\tilde{N}$ as in (46).

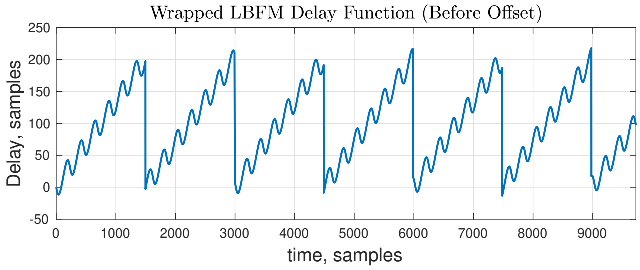

Figure 4: The wrapped LBFM delay function (56) shows negative values and thus the need for an additional offset to ensure it is positive according to (28).

3.3. LBFM using a TVDL with a Wrapped Delay Function

If the desired instantaneous frequency is the LBFM expression in (10), then a sample-by-sample implementation of the TVDL delay function is made more difficult because both terms have a linear component when integrated with respect to time. While the frequency may be integrated numerically within a sample loop to obtain instantaneous phase, substituting into (34) will yield a delay function that is not bounded by a delay line maximum as in (28). Furthermore, the expression for the delay function is not in a form whereby n may be substituted by $\hat{n}$ and the wrapping described by (47) may not be easily applied. It is preferable, therefore, to use the known phase expression (19) of the alternate representation of the LBFM oscillator (18) which, when substituted into (34) and replacing n with $\hat{n}$, yields the LBFM delay function,

$$\hat{D}_b = n - \theta_b \frac{f_s}{\omega_c} = \hat{n}\left(1 - \frac{\omega_0}{\omega_c}\right) + 2\phi(n)\frac{f_s}{\omega_c}. \tag{56}$$

Because $\phi(n)$ in (56), given by (20), is an oscillator with no linear term, it is naturally bounded by upper and lower limits and does not need to be wrapped by (47) as does the linear term $\hat{n}\,(1 - \omega_0/\omega_c)$ in (56).

A final matter that must be addressed before (56) will satisfy (28) and is suitable for use in a TVDL is to ensure the delay function is positive. As shown in Figure 4, the wrapped LBFM delay function (56) possible results in some negative values.

While ensuring a positive function was accomplished in the FM case described in (3.1) simply by introducing an offset (manifesting in the TVDL as a pure delay), a little more care must be taken in the feedback case where introducing an initial phase term other than an integer multiple of 2π will alter the shape of the TVDL output waveform. If an offset d_0 is added to the delay function to make it positive, the output of the TVDL becomes:

$$e^{j\omega_c(n - (D(n) + d_0))T} = e^{j(\omega_c(n - D(n))T - d_0\omega_c T)} \tag{57}$$

and if the initial phase term is an integer k multiple of 2π then

$$d_0 \omega_c T = k2\pi \longrightarrow d_0 = k\frac{2\pi}{\omega_c T} = k\frac{f_s}{f_c}. \tag{58}$$

In most cases, it is sufficient to set $k = 1$ for the minimum offset d_0 needed to make $\hat{D}_b(n)$ positive and to avoid phase distortion (see Figure 5).

4. CONCLUSIONS

Time-variation (e.g. amplitude, frequency and phase modulation, pitch gliding, spectral fluctuations) of a signal or its frequency components is widely observed in musical acoustic systems, often in the presence of nonlinearities [14] and has thus been a topic of significant interest in physical modeling synthesis [15, 16]. TVDLs are well suited for time-warping an input signal with a delay function and have been used effectively to implement phase and frequency modulation in delay-based effects such as flanging and vibrato [17, 18], the Doppler effect [13] as well as to extending waveguide synthesis in strings [19]. Implementing LBFM using a TVDL allows for greater exploration of its possible applications and, in particular, to delay-based synthesis or audio processing where nonlinear effects may be desirable.

In this work, prior LBFM theory is summarized to the extent that it informs the current work of implementing LBFM using a TVDL. Caveats in finding a suitable bounded delay function are presented with solutions that are informed by a previously presented closed-form solution of LBFM that reveals the analytic phase and sounding frequency.

To better understand the difficulty in implementing LBFM using a TVDL, three incremental cases are presented. First, it is shown that using a TVDL to implement regular FM is ideal because the oscillating delay function may be bounded by a minimum that ensures it is positive and by a maximum that sets the size of a circular delay line. Second, it is shown that, because LBFM undergoes a sustained pitch shift when either the feedback coefficient B or carrier frequency ω_c is changed, the corresponding delay function has a component that increases (or decreases depending on the direction of trasponsition) linearly in time, making it unbounded over the duration of the input signal. When the target frequency is known however, as it is from the closed-form solution, the delay function may be wrapped so that, even at discontinuities, the output's phase remains continuous and no artifacts are introduced in the delay line output.

There are three main contributions to this work. The first is the formulation of a TVDL delay function for LBFM using the analytic phase and sounding frequency derived from a previously presented closed-form solution. The second is using the same phase and frequency to construct a wrapped delay function so that it is bounded and may be used to implement a circular delay line. Finally, the third contribution is the careful calculation of an offset that ensures the wrapped delay function is positive while avoiding phase distortion in the TVDL output.

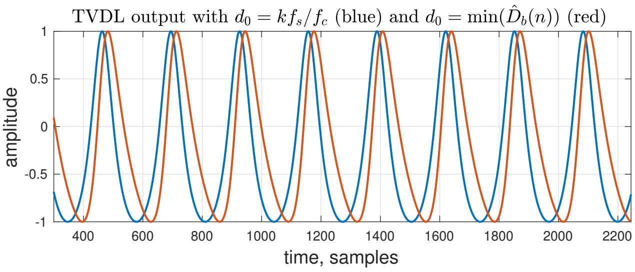

Figure 5: The TVDL output when the offset is set according to (58) (blue) and when simply set to the minimum value required to make (56) positive (red), resulting in phase distortion and a noticeable change in the waveform's shape.

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