STABILITY ANALYSIS OF TIME-VARYING VIRTUAL ANALOG FILTERS

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ABSTRACT

Time-varying virtual analog filters used in digital audio effects and synthesizers are often implemented by discretizing continuous-time state-space systems using trapezoidal integration. When filter parameters such as cutoff frequency or resonance vary over time, as is the norm in musical applications, proving BIBO stability of the resulting time-varying discrete-time system becomes nontrivial. In this paper, we review the technique of common quadratic Lyapunov functions (CQLFs) from the control systems literature and show how a continuous-time CQLF is preserved through discretization. This allows us to prove stability of some time-varying virtual analog filters by working in the often simpler continuous-time domain. We apply this framework to several filters of musical interest, providing new proofs of stability for the state variable filter and the Sallen-Key filter, and new bounds on the stable time-varying parameter range for the Moog ladder and diode ladder filters.

1. INTRODUCTION

Time-varying filters are central to digital audio effects and synthesizers. In actual musical use, parameters such as cutoff frequency and resonance are routinely modulated by envelopes, LFOs, digital audio workstation automation, and direct operator control. Results about stability that depend on the condition of time-invariance of the system cannot be applied in this setting. Stability is nevertheless a key concern for musical applications, as unbounded output could cause failures by overloading downstream systems or causing audible glitches.

Proving stability of time-varying filters is more challenging than the time-invariant case. Linear time-invariant (LTI) filters can be fully characterized by a transfer function, which can be analyzed to easily determine stability of the filter: causal LTI filters are stable if and only if their transfer function's poles lie strictly inside the unit circle. In the time-varying case, filters are not characterized by a single transfer function; there is no simple necessary and sufficient stability criterion. It is well known that parameterized time-varying filters can be unstable even if the system would be LTI stable if the parameters were held constant at any moment [1].

This paper draws on common quadratic Lyapunov function methods from the control systems literature to study this problem. After reviewing some basic definitions, we show in Lemma 2.1 that these methods are equivalent to matrix-norm-based approaches that have previously been applied to audio filters in [1], [2].

The advantage of the Lyapunov viewpoint comes into focus when we consider virtual analog filters that have been derived by discretizing continuous-time filters using trapezoidal integration. Trapezoidal integration has become a popular approach for designing virtual analog filters [3] [4], as it is systematic to apply and yields a frequency response with a close connection to the continuous-time prototype. Theorem 3.1 shows that we can prove results about the stability of the discretized filter by studying Lyapunov functions of the continuous-time prototype system, which is often simpler than the discretized system. This result is a version of a classic theorem [5] from the control systems literature, adapted to our time-varying virtual analog setting.

We then apply this framework to several filters of musical interest in Section 4, including virtual analog filters commonly used in synthesizer applications. We provide proofs of stability for the state variable filter and Sallen-Key filter, as well as the Moog ladder filter and diode ladder filter under restricted parameter ranges. We also exhibit counterexamples for the Moog and diode ladder filters, showing these restricted parameter ranges are close to optimal. Throughout, the discrete-time filters remain our objects of interest; the continuous-time prototypes serve as analytical devices for proving stability.

2. DISCRETE-TIME STATE-SPACE FILTERS

For simplicity, we limit our discussion to single-input, single-output filters. We can characterize such a linear, time-varying filter with inputs $\mathbf{u}[n] \in \mathbb{R}^1$, internal states $\mathbf{x}[n] \in \mathbb{R}^N$, and outputs $\mathbf{y}[n] \in \mathbb{R}^1$ by

$$\begin{aligned} \mathbf{x}[n+1] &= \mathbf{A}[n]\mathbf{x}[n] + \mathbf{B}[n]\mathbf{u}[n], \\ \mathbf{y}[n] &= \mathbf{C}[n]\mathbf{x}[n] + \mathbf{D}[n]\mathbf{u}[n]. \end{aligned} \tag{1}$$

Here $\mathbf{A}[n] \in \mathbb{R}^{N\times N}$, $\mathbf{B}[n] \in \mathbb{R}^{N\times 1}$, $\mathbf{C}[n] \in \mathbb{R}^{1\times N}$, and $\mathbf{D}[n] \in \mathbb{R}^{1\times 1}$ are time-varying matrices representing the operation of the filter.

2.1. BIBO Stability

This paper studies the BIBO (Bounded-input, Bounded-output) stability of various time-varying systems. We say a system of the form (1) is BIBO stable if there exists a constant $G \geq 0$ such that whenever $\mathbf{u}[n]$ is bounded by some $M \geq |\mathbf{u}[n]|$ for all n, we have $|\mathbf{y}[n]| \leq GM$ for all n.

In musical applications, BIBO stability is often the practically relevant notion of stability: audio effects are expected to operate correctly under arbitrary bounded input signals and time-varying user-controlled parameters. A well-behaved effect should not generate unbounded output from bounded excitation.

2.2. Laroche's Stability Criteria

In [1], Laroche provides two convenient criteria for proving the BIBO stability of state-space filters under time-varying conditions. These criteria and extensions of them have previously been used to prove stability of virtual analog filters in [2], [6]. For a time-varying state matrix $\mathbf{A}[n]$, criterion 2 applies if there exist an invertible matrix $\mathbf{T}$ and a constant $\gamma \in [0, 1)$ such that

$$\|\mathbf{T}\mathbf{A}[n]\mathbf{T}^{-1}\| \leq \gamma, \tag{2}$$

for all $n \in \mathbb{N}$. In other words, if there is a single similarity transform under which every state transition is a strict contraction, then Laroche's result, together with bounds on $\mathbf{B}[n]$, $\mathbf{C}[n]$ and $\mathbf{D}[n]$, gives BIBO stability.

Here and in the rest of this paper, the matrix norm $\|\mathbf{M}\|$ of a matrix $\mathbf{M}$ is taken to be the induced Euclidean norm, that is, the maximum amplification that the matrix $\mathbf{M}$ can apply to a vector $\mathbf{x}$:

$$\|\mathbf{M}\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{M}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

2.3. Common Quadratic Lyapunov Functions

The Lyapunov function method is a powerful technique for analyzing stability for time-varying discrete systems, and is widely used in control theory.

We call a symmetric and positive definite matrix $\mathbf{P} \in \mathbb{R}^{N\times N}$ a common quadratic Lyapunov function (CQLF) for a state-space filter of the form (1) if there exists a $\rho \in [0, 1)$ such that

$$(\mathbf{A}[n]\mathbf{x})^T\mathbf{P}(\mathbf{A}[n]\mathbf{x}) \leq \rho \cdot \mathbf{x}^T\mathbf{P}\mathbf{x}, \tag{3}$$

for all $\mathbf{x} \in \mathbb{R}^N, n \in \mathbb{N}$.

We now show that the existence of such a CQLF is equivalent to Laroche's criterion 2.

Lemma 2.1. *A time-varying state matrix $\mathbf{A}[n]$ has a common quadratic Lyapunov function if and only if it satisfies Laroche's criterion 2* (2).

Proof. Given a CQLF $\mathbf{P}$ for $\mathbf{A}[n]$, set $\mathbf{T}$ to be the Cholesky decomposition of $\mathbf{P}$, that is, the unique upper triangular matrix with positive diagonal entries such that $\mathbf{P} = \mathbf{T}^T\mathbf{T}$. Such a matrix always exists by a standard result (e.g. Corollary 7.2.9 in [7]). Then for any $\mathbf{x} \in \mathbb{R}^N$, we have

$$\begin{aligned}(\mathbf{A}[n]\mathbf{x})^T\mathbf{P}(\mathbf{A}[n]\mathbf{x}) &= (\mathbf{T}\mathbf{A}[n]\mathbf{x})^T(\mathbf{T}\mathbf{A}[n]\mathbf{x}) \\ &= \|\mathbf{T}\mathbf{A}[n]\mathbf{x}\|^2.\end{aligned}$$

Similarly, $\mathbf{x}^T\mathbf{P}\mathbf{x} = \|\mathbf{T}\mathbf{x}\|^2$. Because $\mathbf{P}$ was a CQLF, there exists some $\rho \in [0, 1)$ such that $\|\mathbf{T}\mathbf{A}[n]\mathbf{x}\|^2 \leq \rho\|\mathbf{T}\mathbf{x}\|^2$.

We note that $\mathbf{T}$ is invertible (its diagonal entries are all positive), thus an arbitrary $\mathbf{z} \in \mathbb{R}^N$ can be written as $\mathbf{z} = \mathbf{T}\mathbf{x}$ for a unique $\mathbf{x} = \mathbf{T}^{-1}\mathbf{z}$. Substituting, $\|\mathbf{T}\mathbf{A}[n]\mathbf{T}^{-1}\mathbf{z}\|^2 \leq \rho\|\mathbf{z}\|^2$. So, by the definition of matrix norm, $\|\mathbf{T}\mathbf{A}[n]\mathbf{T}^{-1}\| \leq \sqrt{\rho} < 1$ which is exactly Laroche's criterion 2 (taking $\gamma = \sqrt{\rho}$).

For the other direction, we take a $\mathbf{T}$ and $\gamma \in [0, 1)$ satisfying criterion 2. Setting $\mathbf{P} = \mathbf{T}^T\mathbf{T}$, we note first that $\mathbf{P}$ is symmetric, since $(\mathbf{T}^T\mathbf{T})^T = \mathbf{T}^T\mathbf{T}$. $\mathbf{P}$ is also positive definite since for any nonzero $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{x}^T\mathbf{T}^T\mathbf{T}\mathbf{x} = \|\mathbf{T}\mathbf{x}\|^2 > 0$ (criterion 2 implies $\mathbf{T}$ has an inverse, so $\mathbf{T}\mathbf{x} = \mathbf{0}$ only for $\mathbf{x} = \mathbf{0}$).

Then consider, for arbitrary $\mathbf{x} \in \mathbb{R}^N$,

$$\begin{aligned}(\mathbf{A}[n]\mathbf{x})^T\mathbf{P}(\mathbf{A}[n]\mathbf{x}) &= \|\mathbf{T}\mathbf{A}[n]\mathbf{x}\|^2 \\ &= \|\mathbf{T}\mathbf{A}[n]\mathbf{T}^{-1}\mathbf{T}\mathbf{x}\|^2 \\ &\leq \|\mathbf{T}\mathbf{A}[n]\mathbf{T}^{-1}\|^2 \cdot \|\mathbf{T}\mathbf{x}\|^2,\end{aligned}$$

the last inequality coming from the consistency of the matrix norm. But since $\mathbf{T}$ satisfies criterion 2, $\|\mathbf{T}\mathbf{A}[n]\mathbf{T}^{-1}\| \leq \gamma$. So we have

$$\begin{aligned}(\mathbf{A}[n]\mathbf{x})^T\mathbf{P}(\mathbf{A}[n]\mathbf{x}) &\leq \gamma^2\|\mathbf{T}\mathbf{x}\|^2 \\ &= \gamma^2 \cdot \mathbf{x}^T\mathbf{P}\mathbf{x},\end{aligned}$$

which is exactly the statement (3) that $\mathbf{P}$ is a CQLF, with $\rho = \gamma^2$. □

Theorem 2.2. *If a common quadratic Lyapunov function $\mathbf{P}$ exists for $\mathbf{A}[n]$ and we have a finite upper bound G on the norms of $\mathbf{B}[n]$, $\mathbf{C}[n]$, and $\mathbf{D}[n]$, then the filter is BIBO stable.*

Proof. This follows directly from Laroche's results on BIBO stability in [1] and Lemma 2.1. We note that this is a well-known result in control systems literature, with different proofs available in e.g., [8]. □

It is no easier in general to find a CQLF for a filter than it is to directly satisfy Laroche's criterion 2. However, this theorem has value in the case where the filter has been discretized from a continuous-time filter using trapezoidal integration: in that setting, we can derive a CQLF directly from the continuous-time system matrices, which often have a simpler form than the discretized matrices. The next section will state this fact as Theorem 3.1.

3. CONTINUOUS-TIME STATE-SPACE SYSTEMS

Virtual analog and physically modeled filters are commonly designed by discretizing a linear continuous state-space system, often derived by approximating the behavior of an analog circuit or other physical system.

We can characterize a continuous-time linear system with state $\mathbf{x}(t) \in \mathbb{R}^N$, input $\mathbf{u}(t) \in \mathbb{R}^1$, and output $\mathbf{y}(t) \in \mathbb{R}^1$ as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A_C}\mathbf{x}(t) + \mathbf{B_C}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C_C}\mathbf{x}(t) + \mathbf{D_C}\mathbf{u}(t),\end{aligned} \tag{4}$$

with $\mathbf{A_C} \in \mathbb{R}^{N\times N}$, $\mathbf{B_C} \in \mathbb{R}^{N\times 1}$, $\mathbf{C_C} \in \mathbb{R}^{1\times N}$, and $\mathbf{D_C} \in \mathbb{R}^{1\times 1}$.

There is a simple formula for the transfer function $H(s)$ in terms of these state-space matrices:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C_C}\,(sI - \mathbf{A_C})^{-1}\mathbf{B_C} + \mathbf{D_C}. \tag{5}$$

3.1. Discretization Using Trapezoidal Integration

A popular discretization scheme is *trapezoidal integration* [3] [4], also commonly known as the (state-space) bilinear transform or Tustin transform [9]. We can also understand this technique as a linear special case of the K-method described in [10].

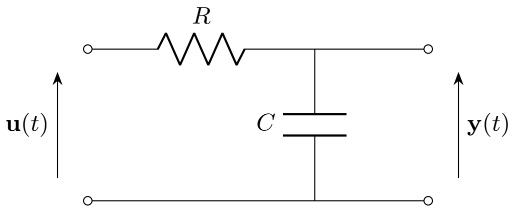

Figure 1: RC low-pass filter circuit with input voltage $\mathbf{u}(t)$ and output voltage $\mathbf{y}(t)$ across the capacitor.

For a linear continuous-time state-space system described by (4), the trapezoidally integrated discrete-time state-space system is given by:

$$\begin{cases} \mathbf{A} = (\mathbf{I} - g\mathbf{A_C})^{-1}(\mathbf{I} + g\mathbf{A_C}), \\ \mathbf{B} = 2g\,(\mathbf{I} - g\mathbf{A_C})^{-1}\mathbf{B_C}, \\ \mathbf{C} = \mathbf{C_C}(\mathbf{I} - g\mathbf{A_C})^{-1}, \\ \mathbf{D} = \mathbf{D_C} + g\,\mathbf{C_C}(\mathbf{I} - g\mathbf{A_C})^{-1}\mathbf{B_C}, \end{cases} \tag{6}$$

where $g > 0$ is a discretization parameter.

This formula is standard in control systems; see, e.g. Theorem 11.2 in [9] for an equivalent formulation using a different scaling. Our formula can be derived by considering the trapezoidal rule

$$\mathbf{x}[n+1] - \mathbf{x}[n] = g\big((\mathbf{A_C}\mathbf{x}[n+1] + \mathbf{B_C}\mathbf{u}[n+1]) + (\mathbf{A_C}\mathbf{x}[n] + \mathbf{B_C}\mathbf{u}[n])\big).$$

(6) follows by rearranging and a change of state variable $\tilde{\mathbf{x}}[n] = (\mathbf{I} - g\mathbf{A_C})\mathbf{x}[n] - g\mathbf{B_C}\mathbf{u}[n]$ which absorbs the implicit $u[n+1]$ term.

Without pre-warping, the discretization parameter g is given by $g = \frac{T}{2}$ where T is the sampling period. We'll often make use of a pre-warping term to map a frequency of interest of the continuous system (such as the cutoff frequency of a low-pass filter) ω_C to a target frequency ω_D of the discrete-time system, in this case $g = \frac{\tan(\omega_D T/2)}{\omega_C}$. For the case studies in this paper, we will normalize the continuous-time filter to have $\omega_C = 1$ radian per second, in this case

$$g = \tan\left(\omega_D \frac{T}{2}\right). \tag{7}$$

So far in this section, we have been considering a time-invariant continuous-time system discretized into a time-invariant discrete-time system. We now introduce time-variation: for each sample n, we discretize a possibly different LTI continuous-time system $\mathbf{A_C}[n]$, $\mathbf{B_C}[n]$, $\mathbf{C_C}[n]$, $\mathbf{D_C}[n]$ to obtain the time-varying discrete-time system $\mathbf{A}[n]$, $\mathbf{B}[n]$, $\mathbf{C}[n]$, $\mathbf{D}[n]$ using the above formulas (6), with potentially time-varying discretization parameter $g[n]$. We emphasize that $[n]$ indexes a sequence of frozen LTI continuous-time systems; the resulting discrete-time system is treated as a switched system, not as the discretization of a single time-varying continuous-time system.

3.2. An Example: One-Pole Low-Pass Filter

To show how state-space discretization works, we will consider a simple example: a one-pole low-pass filter. We can think of this as a linear virtual analog model of a simple RC filter with voltage input $\mathbf{u}(t)$ and voltage output $\mathbf{y}(t)$ (Figure 1). The state variable will be the voltage across the capacitor, $\mathbf{x}(t)$. Then the current is

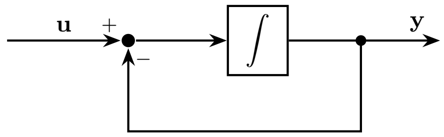

Figure 2: State-space block diagram of the normalized one-pole low-pass filter (8). The block labeled $\int$ denotes integration.

$\mathbf{I}(t) = \frac{\mathbf{u}(t)-\mathbf{x}(t)}{R}$ by Ohm's law and $\frac{\mathbf{I}(t)}{C} = \dot{\mathbf{x}}(t)$ by the capacitor equation. Combining these equations yields the state-space system equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -\frac{1}{RC}\mathbf{x}(t) + \frac{1}{RC}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{x}(t). \end{aligned}$$

Setting $RC = 1$ normalizes the cutoff of the continuous-time filter to 1 radian per second.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -\mathbf{x}(t) + \mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{x}(t). \end{aligned} \tag{8}$$

We can represent this diagrammatically in Figure 2. Reading off the equations, we see $\mathbf{A_C} = -1$, $\mathbf{B_C} = 1$, $\mathbf{C_C} = 1$ and $\mathbf{D_C} = 0$. We now discretize this filter with (6) to yield discrete-time state-space equations

$$\begin{aligned} \mathbf{x}[n+1] &= \frac{1-g[n]}{1+g[n]}\mathbf{x}[n] + \frac{2g[n]}{1+g[n]}\mathbf{u}[n], \\ \mathbf{y}[n] &= \frac{1}{1+g[n]}\mathbf{x}[n] + \frac{g[n]}{1+g[n]}\mathbf{u}[n], \end{aligned} \tag{9}$$

where $g[n]$ is related to the instantaneous cutoff of the discrete-time filter by (7).

3.3. Continuous-Time Common Lyapunov Functions

We call a symmetric and positive definite matrix $\mathbf{P}$ a common quadratic Lyapunov function for a set of continuous-time systems $\mathbf{A_C}[n]$ when there exists $\alpha > 0$ such that

$$\mathbf{x}^T(\mathbf{A_C}[n]^T\mathbf{P} + \mathbf{P}\mathbf{A_C}[n])\mathbf{x} \le -\alpha \cdot \mathbf{x}^T\mathbf{P}\mathbf{x}, \tag{10}$$

for all n and for all $\mathbf{x} \in \mathbb{R}^N$.

We note that this is equivalent to the statement that $\mathbf{A_C}[n]^T\mathbf{P} + \mathbf{P}\mathbf{A_C}[n] + \alpha\mathbf{P}$ is negative semi-definite for all n. Recall that symmetric matrices are negative semi-definite when they have all eigenvalues ≤ 0.

Minimal filter realizations are those where the state dimension equals the number of transfer-function poles. By a standard result, minimal realizations of LTI filters are BIBO stable if and only if they have a quadratic Lyapunov function. This follows, for example, by combining the results in [11, Ch. 6, Thms. 7.2, 7.5, and 9.4]. The filter realizations considered in this paper are all minimal.

3.4. From Continuous-Time to Discrete-Time Lyapunov Functions

We now show how to use a continuous-time CQLF to derive a discrete-time one for a time-varying system. The following theorem is the main technical tool used in the remainder of the paper.

A version of this result for a time-invariant system is standard [5], and related preservation results have also been stated for finite families of continuous filters in [12], [13]. To better suit the setting of parameterized virtual analog systems, we adapt this result to a possibly infinite family of matrices $\mathbf{A_C}[n]$ and time-varying $g[n]$. We provide a direct proof of the specialized result we need.

Theorem 3.1. *When a set of continuous-time systems $\mathbf{A_C}[n]$, with some finite bound M such that for all n, $\|\mathbf{A_C}[n]\| < M$, is discretized into a discrete time-varying system $\mathbf{A}[n]$ using (6) with varying $g[n] \in [g_{\min}, g_{\max})$ with $g_{\min} > 0$, a common quadratic Lyapunov function $\mathbf{P}$ for the continuous-time $\mathbf{A_C}[n]$ is also a common quadratic Lyapunov function for the discrete-time $\mathbf{A}[n]$, and thus the discretized filter is stable so long as $\mathbf{B_C}[n]$, $\mathbf{C_C}[n]$ and $\mathbf{D_C}[n]$ are bounded by a constant independent of n.*

Proof. First solve the expression for $\mathbf{A}$ in (6) for $\mathbf{A_C}[n]$ to get

$$\mathbf{A_C}[n] = \frac{1}{g[n]}(\mathbf{A}[n] - \mathbf{I})(\mathbf{A}[n] + \mathbf{I})^{-1}.$$

We then substitute this into (10), which we have by the assumption that $\mathbf{P}$ is a Lyapunov function for $\mathbf{A_C}[n]$. After rearranging, this substitution yields

$$\mathbf{x}^T(\mathbf{A}[n]^T\mathbf{P}\mathbf{A}[n] - \mathbf{P})\mathbf{x} \leq -\frac{g[n]\alpha}{2}\mathbf{x}^T(\mathbf{A}[n] + \mathbf{I})^T\mathbf{P}(\mathbf{A}[n] + \mathbf{I})\mathbf{x}. \quad (11)$$

Next, note that by adding $\mathbf{I}$ to both sides of the expression for $\mathbf{A}$ in (6) and re-arranging we have

$$\mathbf{A}[n] + \mathbf{I} = 2(\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}.$$

Substituting into (11) we get

$$\begin{aligned}\mathbf{x}^T(\mathbf{A}[n]^T\mathbf{P}\mathbf{A}[n] - \mathbf{P})\mathbf{x} &\leq -2g[n]\alpha((\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}\mathbf{x})^T \\ &\quad \cdot \mathbf{P}(\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}\mathbf{x}.\end{aligned}$$

Since $\mathbf{P}$ is symmetric positive definite, for all $\mathbf{z}$, $\lambda_{\min}(\mathbf{P})\|\mathbf{z}\|^2 \leq \mathbf{z}^T\mathbf{P}\mathbf{z} \leq \lambda_{\max}(\mathbf{P})\|\mathbf{z}\|^2$ (Theorem 4.2.2(c) in [7]). Thus

$$\mathbf{x}^T(\mathbf{A}[n]^T\mathbf{P}\mathbf{A}[n] - \mathbf{P})\mathbf{x} \leq -2g_{\min}\alpha\lambda_{\min}(\mathbf{P})\|(\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}\mathbf{x}\|^2. \quad (12)$$

Note $\mathbf{x} = (\mathbf{I} - g[n]\mathbf{A_C}[n])(\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}\mathbf{x}$ so

$$\begin{aligned}\|\mathbf{x}\| &= \|(\mathbf{I} - g[n]\mathbf{A_C}[n])(\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}\mathbf{x}\| \\ &\leq \|(\mathbf{I} - g[n]\mathbf{A_C}[n])\|\|(\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}\mathbf{x}\| \\ &\leq (1 + g_{\max}M)\|(\mathbf{I} - g[n]\mathbf{A_C}[n])^{-1}\mathbf{x}\|,\end{aligned}$$

where the previous two inequalities use consistency of norms and the triangle inequality with the bounds of $g[n]$ and $\|\mathbf{A_C}[n]\|$, respectively.

Applying this to (12) we have

$$\begin{aligned}\mathbf{x}^T(\mathbf{A}[n]^T\mathbf{P}\mathbf{A}[n] - \mathbf{P})\mathbf{x} &\leq -\frac{2g_{\min}\alpha\lambda_{\min}(\mathbf{P})}{(1 + g_{\max}M)^2}\|\mathbf{x}\|^2 \\ &\leq -\frac{2g_{\min}\alpha\lambda_{\min}(\mathbf{P})}{\lambda_{\max}(\mathbf{P})(1 + g_{\max}M)^2}\mathbf{x}^T\mathbf{P}\mathbf{x}.\end{aligned}$$

Finally, rearranging, we have

$$\mathbf{x}^T\mathbf{A}[n]^T\mathbf{P}\mathbf{A}[n]\mathbf{x} \leq \left(1 - \frac{2g_{\min}\alpha\lambda_{\min}(\mathbf{P})}{\lambda_{\max}(\mathbf{P})(1 + g_{\max}M)^2}\right)\mathbf{x}^T\mathbf{P}\mathbf{x}.$$

Note that the term subtracted from 1 must be positive, so there must exist some $\rho \in [0, 1)$ such that $\mathbf{x}^T\mathbf{A}[n]^T\mathbf{P}\mathbf{A}[n]\mathbf{x} \leq \rho\,\mathbf{x}^T\mathbf{P}\mathbf{x}$, so (3) is satisfied and $\mathbf{P}$ is indeed a quadratic Lyapunov function for the discrete-time system. Stability then follows directly from Theorem 2.2, since (6) will yield bounded $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ given we have a uniform bound on $\mathbf{B_C}$, $\mathbf{C_C}$, and $\mathbf{D_C}$. □

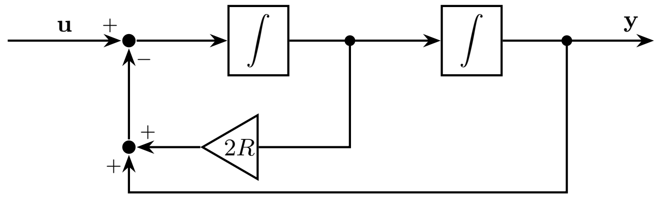

Figure 3: State-space block diagram of the state variable filter (13). The parameter R controls the resonance.

We emphasize that the conclusion of Theorem 3.1 holds in the case where $\mathbf{A_C}[n]$ and $g[n]$ are time-varying, assuming the CQLF $\mathbf{P}$ is common to all $\mathbf{A_C}[n]$.

4. CASE STUDIES

The rest of the paper will explore using these techniques to prove stability results on specific linear time-varying filters. In each case where $g[n]$ varies with time, we assume that there are bounds such that $g[n] \in [g_{\min}, g_{\max})$ where $g_{\min} > 0$.

4.1. Filter Sweeps

A common technique in audio effects and synthesizers is the so-called "filter sweep" or "filter modulation" where the cutoff frequency (or some other frequency of interest) is changed over time.

Many filters are stable if the only parameter that is changing with time is the cutoff frequency of the filter, as the following corollary makes precise.

Corollary 4.1. *A discrete-time filter obtained by discretizing a stable continuous-time LTI filter with a minimal realization via (6), with time-varying $g[n] \in [g_{\min}, g_{\max})$ and $g_{\min} > 0$, is BIBO stable.*

Proof. This is immediate from Theorem 3.1. In this case $\mathbf{A_C}$ is not time-varying. Since the continuous-time system is LTI, all matrices are trivially bounded. □

We note that Corollary 4.1 shows immediately that the one-pole filter we explored in Section 3.2 is stable under time-varying $g[n]$. Laroche's criterion 2 (or even the weaker criterion 1) from [1] can also directly prove stability of this simple filter.

4.2. State Variable Filter

The state variable filter (SVF) is a common analog filter design for musical effects and synthesizers, notably used in the Oberheim SEM. The design features two analog integrators, connected through two different feedback paths as shown in Figure 3. Advantages of the design include relatively easy voltage control for time-variation, as well as the ability to tap the circuit at different points to achieve different output responses (e.g., low-pass vs high-pass).

Discretized versions are popular as digital time-varying filters, either in trapezoidally integrated form ([2], [4]) or in the Chamberlin form ([14], [15]). Advantages of the digital SVF are good time-varying properties (such as stability, as we will show), as well as the ability to produce arbitrary frequency responses as described in [2].

The normalized continuous-time system (for a low-pass filter) is described by:

$$\mathbf{A_C} = \left[\begin{smallmatrix} -2R & -1 \\ 1 & 0 \end{smallmatrix}\right], \quad \mathbf{B_C} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \quad \mathbf{C_C} = \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right], \quad \mathbf{D_C} = 0. \tag{13}$$

Here R represents the gain of one of the feedback paths. Lower values of R represent higher values of resonance. A brief calculation of the transfer function shows that the filter is LTI stable in the case where $R > 0$.

We discretize via (6). Omitting the $[n]$ index and writing $\Delta = 1 + 2gR + g^2$, we obtain

$$\mathbf{A} = \frac{1}{\Delta}\begin{bmatrix} 1 - 2gR - g^2 & -2g \\ 2g & 1 + 2gR - g^2 \end{bmatrix}, \quad \mathbf{B} = \frac{2g}{\Delta}\begin{bmatrix} 1 \\ g \end{bmatrix},$$

$$\mathbf{C} = \frac{1}{\Delta}\begin{bmatrix} g & 1 + 2gR \end{bmatrix}, \quad \mathbf{D} = \frac{g^2}{\Delta}.$$

The continuous-time filter is minimal, since it is a two-pole filter and there are two states, and is stable when $R > 0$. We see immediately from Corollary 4.1 that when R is fixed, the time-varying discrete filter with varying g is BIBO stable.

Stability under arbitrary variation of g and R was proved with Laroche's criterion 2 in [2] and with a multi-step extension of criterion 1 in [6], but both these proofs were complex and relied on computer algebra systems for key steps. Here we present a simpler proof based on Theorem 3.1.

Proposition 4.2. *The discretized state variable filter with time-varying $R[n] \in [R_{\min}, R_{\max}]$ where $R_{\min} > 0$ and time-varying $g[n]$ is BIBO stable.*

Proof. By Theorem 3.1, it suffices to exhibit a CQLF for $\mathbf{A_C}[n]$. Choose any $\varepsilon > 0$ satisfying $\varepsilon < \frac{2R_{\min}}{1+R_{\min}^2}$ and $\varepsilon < \frac{2R_{\max}}{1+R_{\max}^2}$, and set

$$\mathbf{P} = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}.$$

Such an ε exists since both bounds are positive. Since $\frac{2R}{1+R^2} \leq 1$ for all $R > 0$, we have $\varepsilon < 1$ and $\mathbf{P}$ is positive definite. Moreover, since $\frac{2R}{1+R^2}$ is increasing on $(0, 1]$ and decreasing on $[1, \infty)$, the two endpoint conditions imply $\varepsilon < \frac{2R}{1+R^2}$ for all $R \in [R_{\min}, R_{\max}]$. A direct computation gives

$$\mathbf{A_C}^T\mathbf{P} + \mathbf{P}\,\mathbf{A_C} = \begin{bmatrix} -4R + 2\varepsilon & -2R\varepsilon \\ -2R\varepsilon & -2\varepsilon \end{bmatrix}. \tag{14}$$

To prove (10), it is enough to show that the matrix in (14) is negative definite for all $R \in [R_{\min}, R_{\max}]$, since its entries depend continuously on R and compactness then gives a uniform $\alpha > 0$. To do that we use the standard Sylvester criterion (e.g., Theorem 7.2.5 in [7]): $-4R + 2\varepsilon < 0$ since $\varepsilon < 2R/(1 + R^2) < 2R$, and the determinant is $4\varepsilon\big(2R - \varepsilon(1 + R^2)\big) > 0$. So the matrix is negative definite for all $R \in [R_{\min}, R_{\max}]$, meaning $\mathbf{P}$ is a continuous-time CQLF. Since $\mathbf{B_C}$, $\mathbf{C_C}$, and $\mathbf{D_C}$ are constant, Theorem 3.1 gives BIBO stability. □

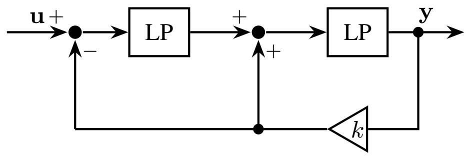

Figure 4: Block diagram of the normalized Sallen-Key filter (15): two cascaded one-pole sections (8) with a feedback gain k.

4.3. Sallen-Key Filter

Another classic analog filter used in effects and synthesizers is the Sallen-Key filter [16], notably used in the Korg MS-20 synthesizer [17]. We can analyze this as two low-pass stages of the form (8), with two feedback paths scaled by a parameter k. We show the design schematically in Figure 4.

$$\begin{aligned} \mathbf{A_C} &= \begin{bmatrix} -1 & -k \\ 1 & k-1 \end{bmatrix}, & \mathbf{B_C} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \mathbf{C_C} &= \begin{bmatrix} 0 & 1 \end{bmatrix}, & \mathbf{D_C} &= 0. \end{aligned} \tag{15}$$

Here k represents the feedback gain, with higher values of k representing higher resonance. A brief calculation of the transfer function shows that this system is LTI stable for $k < 2$.

This system can be discretized with (6). Omitting the $[n]$ index and writing $\Delta = 1 + 2g + g^2 - gk$, we obtain

$$\mathbf{A} = \frac{1}{\Delta}\begin{bmatrix} 1 - g^2 - gk & -2gk \\ 2g & 1 - g^2 + gk \end{bmatrix},$$

$$\mathbf{B} = \frac{2g}{\Delta}\begin{bmatrix} 1 + g - gk \\ g \end{bmatrix},$$

$$\mathbf{C} = \frac{1}{\Delta}\begin{bmatrix} g & 1 + g \end{bmatrix}, \quad \mathbf{D} = \frac{g^2}{\Delta}.$$

We prove stability by means of the following lemma, which can be used to relate CQLFs of similar filter families.

Lemma 4.3. *If $\mathbf{P}$ is a common quadratic Lyapunov function for a family of continuous-time matrices $\mathbf{A_{C1}}[n]$, and there exists a constant $\mathbf{T}$ such that*

$$\mathbf{T}\mathbf{A_{C2}}[n]\mathbf{T}^{-1} = \mathbf{A_{C1}}[n], \tag{16}$$

for a different family $\mathbf{A_{C2}}[n]$, then $\mathbf{T}^T\mathbf{P}\mathbf{T}$ is a common quadratic Lyapunov function for $\mathbf{A_{C2}}[n]$.

Proof. Since $\mathbf{P}$ is a CQLF for $\mathbf{A_{C1}}$, (10) must hold for $\mathbf{A_{C1}}$. Substituting (16) into (10) yields

$$\mathbf{x}^T((\mathbf{T}^{-1})^T\mathbf{A_{C2}}[n]^T\mathbf{T}^T\mathbf{P} + \mathbf{P}\mathbf{T}\mathbf{A_{C2}}[n]\mathbf{T}^{-1})\mathbf{x} \leq -\alpha \cdot \mathbf{x}^T\mathbf{P}\mathbf{x}.$$

Substituting $\mathbf{x} = \mathbf{T}\mathbf{z}$ gives

$$\mathbf{z}^T(\mathbf{A_{C2}}[n]^T\mathbf{T}^T\mathbf{P}\mathbf{T} + \mathbf{T}^T\mathbf{P}\mathbf{T}\mathbf{A_{C2}}[n])\mathbf{z} \leq -\alpha \cdot \mathbf{z}^T\mathbf{T}^T\mathbf{P}\mathbf{T}\mathbf{z}.$$

This shows that $\mathbf{T}^T\mathbf{P}\mathbf{T}$ is a CQLF for $\mathbf{A_{C2}}[n]$. □

Proposition 4.4. *The discretized Sallen-Key filter with time-varying $k[n] \in [0, k_{\max}]$ where $k_{\max} < 2$ and time-varying $g[n]$ is BIBO stable.*

Proof. Consider

$$\mathbf{T} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Applying this to the Sallen-Key $\mathbf{A_C}$ from (15), we have

$$\mathbf{T}\mathbf{A_C}\mathbf{T}^{-1} = \begin{bmatrix} k-2 & -1 \\ 1 & 0 \end{bmatrix},$$

which matches exactly the state variable filter from (13) with $R = 1 - \frac{k}{2}$. Thus since (13) has a CQLF by the proof of Proposition 4.2, we have a CQLF for our Sallen-Key matrix $\mathbf{A_C}$ from (15) by Lemma 4.3, which gives stability by Theorem 3.1 since other state matrices are bounded. $\square$

We emphasize that although the similarity transform $\mathbf{T}$ exists connecting the Sallen-Key filter and the SVF, the time-varying output of the Sallen-Key filter is indeed different from that of the SVF. This can be understood by considering the fact that $\mathbf{T}\mathbf{B_C}$ of the Sallen-Key from (15) is not the same as $\mathbf{B_C}$ of the SVF from (13).

4.4. Moog Ladder Filter

One of the most ubiquitous time-varying filters in effect and synthesizer design is the Moog ladder filter [18], notably used in the popular Minimoog analog synthesizer [19]. In synthesizer applications, the filter is commonly connected to an envelope generator, and thus time-varying.

A circuit analysis and derivation of a linearized approximation is carried out in [20] and [4]. To summarize, we can think of the linearized Moog ladder filter as a cascade of four single-pole sections of the form of (8), with a global feedback. We show this design schematically in Figure 6.

The normalized state-space continuous-time system is:

$$\begin{aligned} \mathbf{A_C} &= \begin{bmatrix} -1 & 0 & 0 & -k \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{B_C} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ \mathbf{C_C} &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D_C} = 0. \end{aligned} \tag{17}$$

Here the parameter k represents the feedback gain of the filter, often called emphasis or resonance in synthesizer applications.

This system can be discretized with (6). Writing $a = 1 + g$, $b = 1 - g$, $\Delta = a^4 + kg^4$, omitting $[n]$ indices, we obtain

$$\mathbf{A} = \frac{1}{\Delta}\begin{bmatrix} ba^3 - kg^4 & -2kg^3 & -2kg^2a & -2kga^2 \\ 2ga^2 & ba^3 - kg^4 & -2kg^3 & -2kg^2a \\ 2g^2a & 2ga^2 & ba^3 - kg^4 & -2kg^3 \\ 2g^3 & 2g^2a & 2ga^2 & ba^3 - kg^4 \end{bmatrix},$$

$$\mathbf{B} = \frac{2g}{\Delta}\begin{bmatrix} a^3 \\ ga^2 \\ g^2a \\ g^3 \end{bmatrix}, \quad \mathbf{C} = \frac{1}{\Delta}\begin{bmatrix} g^3 & g^2a & ga^2 & a^3 \end{bmatrix}, \quad \mathbf{D} = \frac{g^4}{\Delta}.$$

The continuous-time system is stable for $k \in [0, 4)$, so we have immediately from Corollary 4.1:

Proposition 4.5. *The discretized Moog filter is BIBO stable under varying $g[n]$ with fixed $k \in [0, 4)$.*

The time-varying k case is more subtle. A study of time-varying stability of the continuous system using Lyapunov functions was carried out in [20]. In that paper, a CQLF for the continuous-time system in the case where $k \in [0, \frac{5}{3})$ was presented. Here we improve that result with a different function that is common for systems with a wider range, $k \in [0, 2.88]$, using a $\mathbf{P}$ found with a random numerical search.

Proposition 4.6. *The discretized Moog filter with time-varying $k[n] \in [0, 2.88]$ and time-varying $g[n]$ is BIBO stable.*

Proof. Again, by Theorem 3.1, all we need is a CQLF for the continuous-time systems $\mathbf{A_C}$. Here we set

$$\mathbf{P} = \operatorname{diag}(1.14397715, 1.71579237, 2.69245163, 4).$$

Since $\mathbf{A_C}^T\mathbf{P} + \mathbf{P}\mathbf{A_C} + \alpha\mathbf{P}$ is affine in k, to show that the expression is negative definite it suffices to only check it at the endpoints of the range, i.e., 0 and 2.88. Setting $\alpha = 6.8 \times 10^{-6}$, we can approximate the eigenvalues of $\mathbf{A_C}^T\mathbf{P} + \mathbf{P}\mathbf{A_C} + \alpha\mathbf{P}$ numerically. We get

$$\begin{aligned} k = 0: &\quad \{-11.26,\ -5.479,\ -2.364,\ -0.001352\} \\ k = 2.88: &\quad \{-11.80,\ -6.662,\ -0.6379,\ -0.001353\}, \end{aligned}$$

which are all strictly negative. This shows that $\mathbf{P}$ is a common Lyapunov function on this range and that the system is BIBO stable. $\square$

Proposition 4.6 imposes a tighter bound on k than in the fixed-k, varying-g case, where stability held on the wider range $k \in [0, 4)$. In fact, restricting the range of k is necessary:

Proposition 4.7. *The discretized Moog filter with time-varying $k[n] \in [0, 2.89]$ and time-varying $g[n]$ is not BIBO stable in general.*

Proof. We provide a counterexample: taking a $\mathbf{u}[0] = 1$ impulse as input, $\mathbf{u}[n] = 0$ otherwise, and a constant $g[n] = 1.3764$, we vary $k[n]$ with 0 for negative n and the sequence $k[n] = 0, 2.89, 0, 2.89, \ldots$ for positive n.

Let $\mathbf{A}_0$ and $\mathbf{A}_1$ denote the state-transition matrices for $k = 0$ and 2.89, respectively. Note even-sample states after the impulse satisfy $\mathbf{x}[2m+2] = \mathbf{F}\mathbf{x}[2m]$ where $\mathbf{F} = \mathbf{A}_1\mathbf{A}_0$. Numerically, $\mathbf{F}$ has distinct eigenvalues, thus is diagonalizable to $\mathbf{F} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. Defining $\mathbf{z}[m] = \mathbf{V}^{-1}\mathbf{x}[2m]$, we obtain the autonomous LTI system $\mathbf{z}[m+1] = \mathbf{D}\mathbf{z}[m]$. A direct numerical calculation shows the eigenvalue of $\mathbf{F}$ with the largest modulus is $\lambda_1 \approx -1.001$, i.e., $|\lambda_1| > 1$. The corresponding coordinate of the actual initial state $\mathbf{z}[1]$ is non-zero: $z_1[1] \approx -0.72$. Hence $z_1[m] = \lambda_1^{m-1}z_1[1]$ is unbounded. Moreover, the coefficient of $z_1[m]$ in $\mathbf{y}[2m] = \mathbf{C}[2m]\mathbf{V}\mathbf{z}[m]$ is numerically nonzero, so the output $\mathbf{y}$ is also unbounded.

Simulating this example numerically confirms this result, as shown in the upper panel of Figure 5. $\square$

For comparison, replacing 2.89 by 2.88, as in the lower panel of Figure 5, shows that the output remains bounded, consistent with Proposition 4.6.

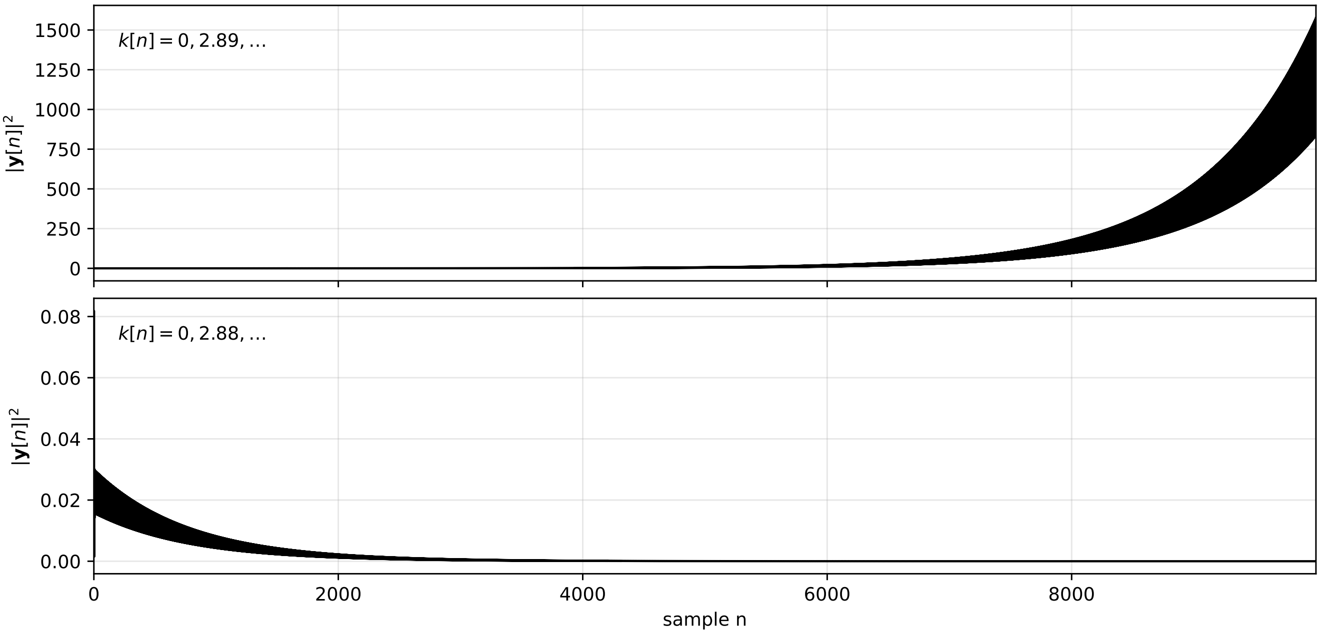

Figure 5: Simulated squared output magnitude $|\mathbf{y}[n]|^2$ for the discretized Moog filter with impulse input $\mathbf{u}[0] = 1$ and constant $g[n] = 1.3764$. The upper panel shows the sequence $k[n] = 0, 2.89, 0, 2.89, \ldots$, for which the output exhibits sustained growth. The lower panel shows the sequence $k[n] = 0, 2.88, 0, 2.88, \ldots$, which remains bounded.

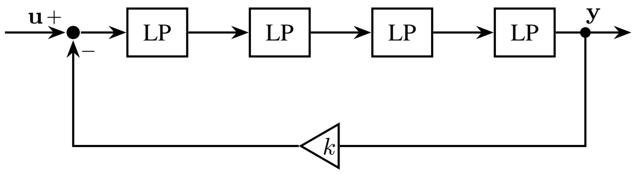

Figure 6: Block diagram of the linearized Moog ladder filter (17): four cascaded one-pole sections (8) with global feedback gain k.

4.5. Diode Ladder Filter

The diode ladder filter topology is a variation on the Moog ladder filter common in synthesizers, notably the Roland TB-303 synthesizer. It differs from the Moog filter in that the stages are *unbuffered*, that is, each filter capacitor receives current not only from the stage's input, but also from the stage's output. Many variants of the analog circuit are common in musical applications, as analyzed in depth in [19]. Here we consider the version with equal capacitance for each filter capacitor. The continuous-time system of such a circuit can be described by:

$$\mathbf{A_C} = \begin{bmatrix} -2 & 1 & 0 & -k \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{B_C} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C_C} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D_C} = 0. \tag{18}$$

Using (5), we can derive the transfer function as

$$H(s) = \frac{1}{s^4 + 7s^3 + 15s^2 + 10s + 1 + k}.$$

This filter can be discretized with (6), but the resulting discrete-time state-space matrices are lengthy, so we omit them for reasons of space.

From the transfer function we can solve the quartic equation for the denominator to show that the filter is stable (i.e., has poles with negative real part) for $k \in [0, \frac{901}{49})$, where the upper bound of the stability range is ≈ 18.4.

Thus, immediately from Corollary 4.1 we have

Proposition 4.8. *The discretized diode ladder filter is BIBO stable under varying* $g[n]$ *with fixed* $k \in [0, \frac{901}{49})$.

Similar to the Moog ladder filter, it turns out that the diode ladder is not time-varying stable under arbitrary modulation of g and k. However, under a restricted range of k, we show stability using a Lyapunov function $\mathbf{P}$ found with a numerical search.

Proposition 4.9. *The discretized diode ladder filter with time-varying* $k[n] \in [0, 8.90]$ *and time-varying* $g[n]$ *is BIBO stable.*

Proof. We follow the same verification plan as Proposition 4.6. First, consider the positive-definite symmetric matrix $\mathbf{P}$ shown in Figure 7. Direct numerical calculation shows that this is a Lyapunov function for $\mathbf{A_C}$ in (18) for $k = 0$ and $k = 8.90$, with $\alpha = 10^{-6}$. By the same endpoint-check argument used in Proposition 4.6, this shows that $\mathbf{P}$ is a CQLF for the whole range. Then, by Theorem 3.1 and the boundedness of the other state matrices, this shows that the discretized filter is time-varying stable. □

Proposition 4.10. *The discretized diode ladder filter with time-varying* $k[n] \in [0, 8.91]$ *and time-varying* $g[n]$ *is not BIBO stable in general.*

$$\mathbf{P} = \begin{bmatrix} 0.025562765630 & 0.042774973981 & 0.037068149991 & -0.014439094598 \\ 0.042774973981 & 0.081070193860 & 0.095883346718 & 0.042487175483 \\ 0.037068149991 & 0.095883346718 & 0.182904686270 & 0.252435242344 \\ -0.014439094598 & 0.042487175483 & 0.252435242344 & 0.710462354240 \end{bmatrix}.$$

Figure 7: Common quadratic Lyapunov matrix used in the verification of time-varying stability of the discretized diode ladder filter over $k \in [0, 8.90]$.

Proof. We provide a counterexample that shows the diode ladder filter is actually unstable in general in this extended parameter range. We set $g = 1.8161$ constant and then set $k = 0, 8.91, 0, 8.91, \dots$ alternating with input an impulse at $\mathbf{u}[0]$. A similar argument to Proposition 4.7 can be used on this example to show that the diode ladder filter is unstable in this extended range. Direct calculation shows that the combined matrix of the 2-period pattern has eigenvalue ≈ -1.0008, with modulus over 1. Numerically, this unstable mode is both excited by the impulse and visible in the output, hence the output is unbounded. $\square$

5. CONCLUSION

We used common quadratic Lyapunov functions to study stability of trapezoidally integrated, time-varying virtual analog filters. Theorem 3.1 shows that this reduces a class of discrete-time BIBO stability questions to continuous-time Lyapunov analysis.

Corollary 4.1 showed that all filters designed by trapezoidally integrating stable continuous-time filters are stable under arbitrary cutoff modulation.

We applied our framework to several filters with varying parameters other than cutoff frequency, yielding general stability results for the SVF and Sallen-Key filters. We also showed that the Moog ladder and diode ladder filters are not stable under arbitrary variation over their full parameter ranges, but are stable on a restricted range.

Heuristic arguments (e.g., in [8]) suggest that even these unstable filters may indeed be stable under the condition of sufficiently slowly changing parameters. Making these intuitions precise in musical contexts where parameters are expected to vary smoothly from sample to sample would be an interesting direction for future work.

Another potential avenue for future research would be exploring modifications to the state-space discretization procedure to ensure time-varying stability for a wider class of filters.

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