# DISPERSION MODELING IN WAVEGUIDE PIANO SYNTHESIS USING TUNABLE ALLPASS FILTERS

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## ABSTRACT

This paper extends a previously proposed method for designing filters simulating the dispersion phenomenon occurring in string instruments. In digital waveguide synthesis, the phenomenon is traditionally modeled by inserting an allpass filter to the string model feedback loop. In this paper, the concept of tunable dispersion filter design, which provides a closed-form formula to design a dispersion filter, is applied to a cascade of first-order allpass filters. Moreover, the method is extended to design a filter cascade including an arbitrary number of first-order filters. In addition, it is shown how the designed dispersion filter can be used in a waveguide piano synthesis model.

## 1. INTRODUCTION

Dispersion is an important phenomenon present in string instruments making produced tones inharmonic. The audibility of the phenomenon depends on the instrument, for example, in the piano it is a perceptually important effect that needs to be taken into account in sound synthesis [1].

In digital waveguide modeling technique [2, 3], dispersion is modeled by inserting an allpass filter simulating the phase delay response of the dispersion phenomenon in the string model [2, 3, 4, 5, 6, 7, 8]. A common way to implement the dispersion filter is to use either a high-order allpass filter [4] or a cascade of low-order filters [5, 6, 7]. An excellent overview of dispersion filter design is given in [8].

Van Duyne and Smith [5] proposed the use of a cascade of first- order filters. In this paper, the idea is extended by introducing a closed-form formula to determine the filter coefficients based on the tunable dispersion filter design method [7].

This paper is organized in the following way. In the beginning, the dispersion phenomenon is introduced in Section 2. Then, the previously proposed tunable dispersion filter design method for a cascade of second-order filters is presented is Section 3, followed by the adaptation for first-order filters in Section 4. The results from the filter design are shown in Section 5 and the conclusions are presented in Section 6.

## 2. DISPERSION PHENOMENON

Dispersion is the property of a string, due to its stiffness, that causes the frequencies of the partials to be higher than the harmonic partials. The frequencies of inharmonic partials can be calculated [1]:

$$f_k = kf_0\sqrt{1+Bk^2}, \tag{1}$$

where $k$ is the partial number, $f_0$ is the nominal fundamental frequency of the ideal string (non-dispersive), and $B$ is the inharmonicity coefficient. The value $B$ can be calculated using string parameters [1]

$$B = \frac{\pi^3 Q d^4}{64 l^2 T}, \tag{2}$$

where $Q$ is Young's modulus, $d$ is the diameter of the string, $l$ is the length, and $T$ is the string tension. Hence, there is a strong relation between the $B$ value and the tension of the string.

Fletcher *et al.* suggested in 1962 that inharmonicity is a significant factor in producing the sound characteristic of the piano [1]. Even though Galembo *et al.* recently proposed that the spectral density might be even a more important factor [9], it is still clear that inharmonicity is an essential property, which needs to be taken into account in piano synthesis.

Dispersion occurs in all string instruments, especially the piano is known to have strong inharmonicity [1]. The amount of inharmonicity depends on the type of the piano; grand pianos tend to have less inharmonicity than upright pianos. Figure 1 depicts estimated $B$ values from a Steinway grand piano. The $B$ values were obtained by an automatic estimation algorithm, which first uses the fast Fourier transform (FFT) to calculate the frequencies of the partials, and then estimates the $B$ value by fitting (1) to the measured data. Some values are missing in Figure 1, because the estimated values for some keys were unreliable (the most common reasons for excluding a $B$ estimate were that the nominal fundamental frequency was estimated incorrectly, or that our peak picking algorithm got confused). The solid lines in the figure show the expected $B$ range for pianos based on our results. Our estimated range is in line with the results by Askenfelt and Galembo [10].

Although the inharmonicity is stronger at the higher end of the range, it has been shown by Järveläinen *et al.* that the inharmonicity is perceived mostly at the lower end of the range [11] (recent work by Järveläinen and Karjalainen suggest, however, that there may be variation in the perception depending on the instrument [12]). The estimated $B$ values, displayed in Figure 1, are all above the threshold of audibility, which, along with the confidence limits, are published in [11]. Thus, it can be expected that inharmonicity is heard through-out the range of the piano.

## 3. TUNABLE DISPERSION FILTER DESIGN METHOD

The tunable dispersion filter design method provided a way to use closed-form formulas to design a dispersion filter [7]. The original design was made for a cascade of second-order filters, but the method can be used to design an arbitrary-order filter cascade.

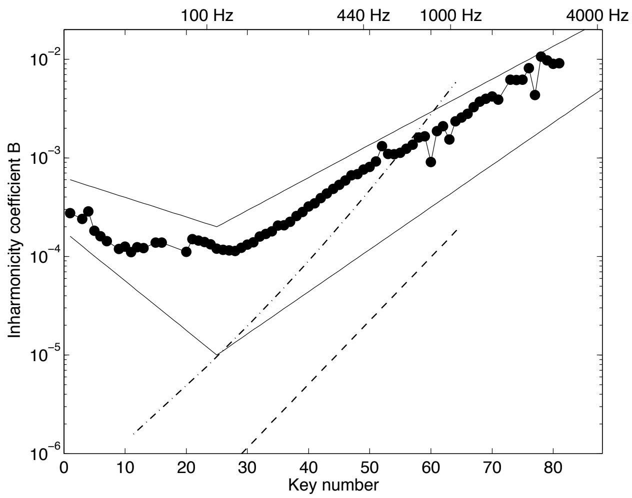


Figure 1: *Measured B values (dots) from a Steinway grand piano*[1]. *The solid lines show the expected B range for the piano, the dashed line shows the threshold of audibility, and the dash-dotted line its confidence limit [11].*

The tunable dispersion filter design method uses the Thiran all-pass filter design to determine the filter coefficients [13, 14, 15]. The Thiran allpass design method is commonly used to design fractional delay filters, but with large delay values the phase delay response becomes similar to the phase delay required to simulate the dispersion.

In the Thiran design method, the filter coefficients are determined by using the desired delay value $D$ at dc. The tunable dispersion filter design method offers a formula to compute $D$ according to the desired $f_0$ and $B$ [7]:

$$D(I_{key}, B) = e^{(C_d(B) - I_{key} k_d(B))}, \tag{3}$$

where

$$I_{key}(f_0) = \log_{\sqrt[12]{2}} \frac{f_0 \sqrt[12]{2}}{27.5}, \tag{4}$$

$$k_d(B) = e^{(k_1 (\ln B)^2 + k_2 \ln B + k_3)}, \tag{5}$$

$$C_d(B) = e^{(C_1 \ln B + C_2)}, \tag{6}$$

and $k_1$, $k_2$, $k_3$, $C_1$, and $C_2$ are predefined constants presented in Table 1. It has been noticed that, when the inharmonicity coefficient value is fixed, the relation between required delay values (on a logarithmic scale) and key indices can be approximated with a straight line. Hence, parameter $k_d$ can be interpreted as the slope coefficient of the line and parameter $C_d$ as the remainder of the line formula. The line parameters $k_d$ and $C_d$ are then parameterized, depending on the inharmonicity coefficient, with parameters $k_1$, $k_2$, $k_3$, $C_1$, and $C_2$.

The dispersion filter, consisting of a cascade of four second-order allpass filters, produces an extra delay that must be taken into account in the string model by modifying the delay line length and the tuning filter coefficients [7]. The extra delay can be estimated by using $D$ value and multiplying it with $M$, the number of filters in cascade.

[1]Steinway grand piano samples from University of Iowa Electronic Music Studios, http://theremin.music.uiowa.edu

| Parameter | Filter 1 ($M = 4$) | Filter 2 ($M = 1$) |
|---|---|---|
| $k_1$ | -0.00050469 | -0.0026580 |
| $k_2$ | -0.0064264 | -0.014811 |
| $k_3$ | -2.8743 | -2.9018 |
| $C_1$ | 0.069618 | 0.071089 |
| $C_2$ | 2.0427 | 2.1074 |

Table 1: *Predefined parameters for the tunable dispersion filter using second-order filters [7]. Filter 1 (four filters in a cascade) is used for key numbers 1-44 and filter 2 (a single filter) for key numbers 45-88.*

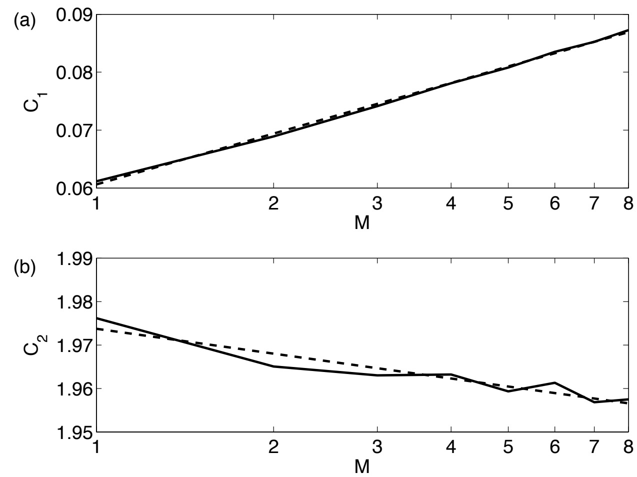


Figure 2: *Searched (a) $C_1$ and (b) $C_2$ values for first-order filters with $M$ values 1 to 8 (solid line). The dashed line illustrates the parameterized values using the straight line fit.*

## 4. NEW FIRST-ORDER DISPERSION FILTER DESIGN

In this paper, a cascade of first-order filters is used for designing a dispersion filter originally proposed by Van Duyne and Smith [5]. A cascade of identical filters is not as effective from the performance point of view as a cascade of filters with different coefficient values. However, the advantage of using a cascade of identical filters is that it is easy to design and, hence, it is used in this work.

The problem with this idea has been the lack of closed-form design formulas, as Van Duyne and Smith did not indicate how the filter cascade can be designed, other than by trial and error or by using a search algorithm [5]. This problem can be solved by extending the tunable dispersion filter method [7]. In this section, the predefined parameters (see equations 3-6) are determined for the tunable dispersion filter method using a cascade of an arbitrary number of first-order allpass filters. Hence, the formula should take the fundamental frequency, the $B$ value, and the number of filters in cascade $M$ as input parameters.

The five predefined constant parameters in equation (5) and (6) were determined for eight cases with $M$ varying from 1 to 8 in a way similar to [7]. It was noticed that parameters $k_1$, $k_2$, and $k_3$ lacked any kind of general trend. However, it seems that the importance of these parameters is minor compared to parameters $C_1$ and $C_2$, which had a clear linear trend on a logarithmic scale, as seen in Figure 2. The explanation for this is that the slope coef-

| Parameter | Value |
|---|---|
| $k_1$ | -0.00179 |
| $k_2$ | -0.0233 |
| $k_3$ | -2.93 |
| $m_1$ | 0.0126 |
| $m_2$ | 0.0606 |
| $m_3$ | -0.00825 |
| $m_4$ | 1.97 |

Table 2: *Calculated filter design parameters for a cascade of first-order Thiran allpass filters.*

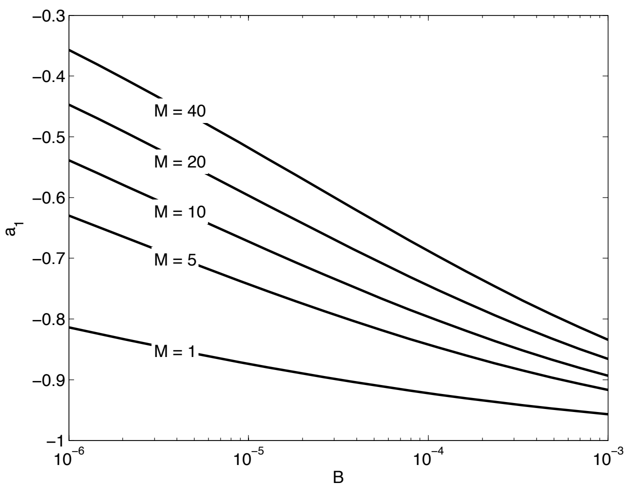


Figure 3: *The dispersion filter coefficient $a_1$ values with $B$ values from $10^{-6}$ to $10^{-3}$ when the number of first-order allpass filters in cascade M is 1, 5, 10, 20, and 40. The fundamental frequency is 65.406 Hz (note $\mathrm{C}_2$) in all cases.*

ficient $k_d$ does not depend on the number of filters in cascade $M$, whereas the remainder term $C_d$ is strongly linked to $M$. Hence, by using polynomial approximation, equation (6) can be extended as

$$C_d(B, M) = e^{((m_1 \ln M + m_2) \ln M + m_3 \ln M + m_4)} \tag{7}$$

where $m_1$, $m_2$, $m_3$, and $m_4$ are the polynomial coefficients defined in Table 2.

The parameters required by the design formulas (3), (5), and (7) were optimized using polynomial approximation. Parameter values $k_1$, $k_2$, and $k_3$ were determined by using $M = 8$. The resulting values are shown in Table 2.

In summary, the whole design process goes as follows. First, the desired $B$, $M$, and $f_0$values should be decided. Then, the required $D$ values can be determined by using equations (3), (4), (5), and (7). Finally, the first-order allpass filter coefficient $a_1$ can be computed by using the Thiran allpass filter method [15]:

$$a_1 = \frac{1 - D}{D + 1} \tag{8}$$

Two examples on how $a_1$ behaves when the inharmonicity coefficient value is changed are shown in Figures 3 and 4.

The delay line length and the tuning filter coefficient should be modified according to the additional delay produced by the dispersion filter. It has to be accounted for that the product $D \cdot M$ may not be a sufficient estimate for the delay with large $M$ values, as suggested in [7] for a cascade of four second-order filters, because the estimation error is approximately $M$ times larger compared to a single allpass filter.

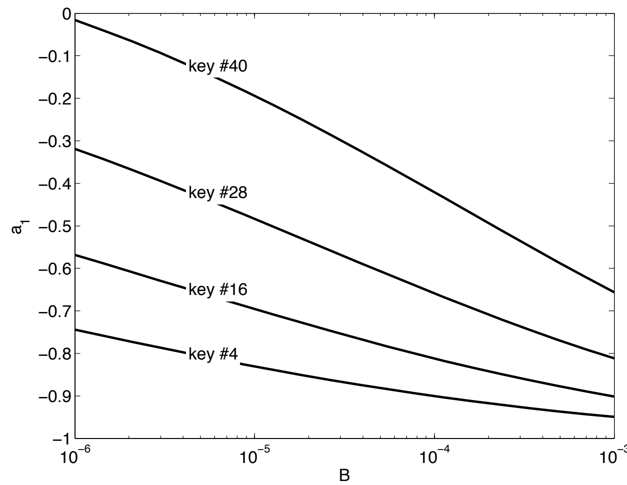


Figure 4: *The dispersion filter coefficient $a_1$ values with $B$ values from $10^{-6}$ to $10^{-3}$ at key indices 4, 16, 28, and 40, which correspond to keys $\mathrm{C}_1$, $\mathrm{C}_2$, $\mathrm{C}_3$, and $\mathrm{C}_4$, respectively. The number of filters in cascade $M$ is 8 in all cases.*

## 5. RESULTS

Figure 5 shows the deviation of the partial frequencies of the example filter proposed in this work at three fundamental frequencies (keys $\mathrm{C}_1$, $\mathrm{C}_2$, and $\mathrm{C}_3$) with reasonable inharmonicity coefficient values, with the values for the number of filters in cascade $M$ set to 1, 5, 10, 20, 30, and 40. The partial frequencies are compared to the target partial frequencies obtained from equation (1). Hence, an ideal dispersion filter would correspond to zero deviation at all frequencies. The figure shows that the quality of the filter improves when $M$ is increased. On the other hand, when the number of filters in cascade $M$ reaches approximately 40, the phase delay response does not fit within the defined error limits. In practical cases, $M$ values below 20 are preferred for computational reasons. Hence, the parameter selection for the filter design is a trade-off between computational load and quality, but the quality does not improve limitlessly. Furthermore, the fixing of parameters $k_1$, $k_2$, and $k_3$ does not add too much bias to the results when $M < 40$, since the responses stay within the defined error limits.

The proposed filter, as well as the original tunable dispersion filter method [7], can be used for providing real-time control over the inharmonicity coefficient. This can be done by recomputing equations (3), (5), (7), and (8), and by updating the filter coefficient each time the inharmonicity coefficient value is changed. The updating process requires eight additions, six multiplications, and one division from the computational point of view. Additionally, the logarithmic function is called once and the exponential function three times during the process.

Finally, the filter was included in a simple piano synthesis model. The number of filters in cascade $M$ was set to 8 in this example. The piano model included a basic string model, as shown

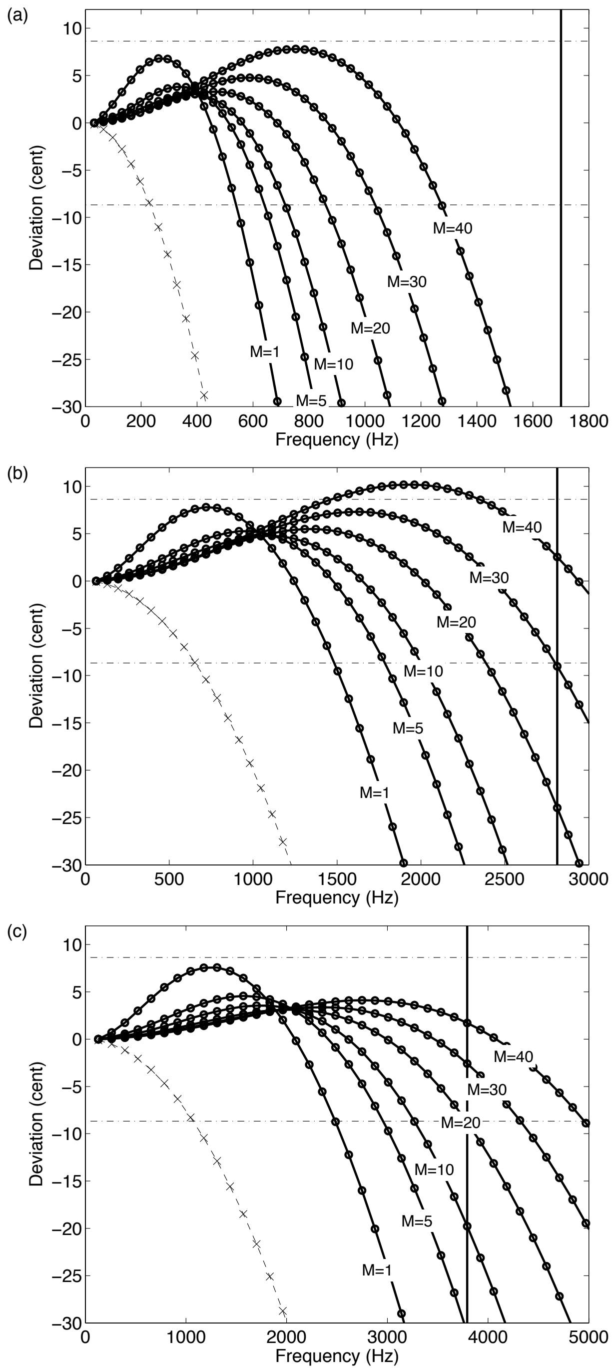


Figure 5: *Deviation of the partial frequencies for the proposed filter (solid line with circles) with the number of filters in cascade* $M$ *values 1, 5, 10, 20, 30, and 40, when (a)* $B = 0.00020$*,* $f_0 = 32.703$ *Hz (key* $C_1$*), (b)* $B = 0.00010$*,* $f_0 = 65.406$ *Hz (key* $C_2$*), and (c)* $B = 0.00015$*,* $f_0 = 130.82$ *Hz (key* $C_3$*). The deviation for a harmonic tone is denoted as a dashed line and the maximum number of perceived inharmonic partials [16] is denoted as a vertical line. The dash-dotted lines are* $0.5\%$ *error limits corresponding to* $\pm 8.39$ *cents.*

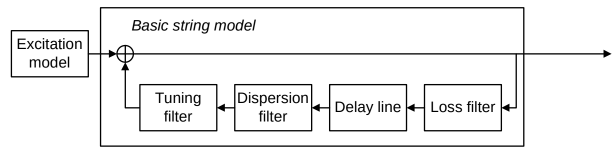


Figure 6: *An illustration of a basic piano string model.*

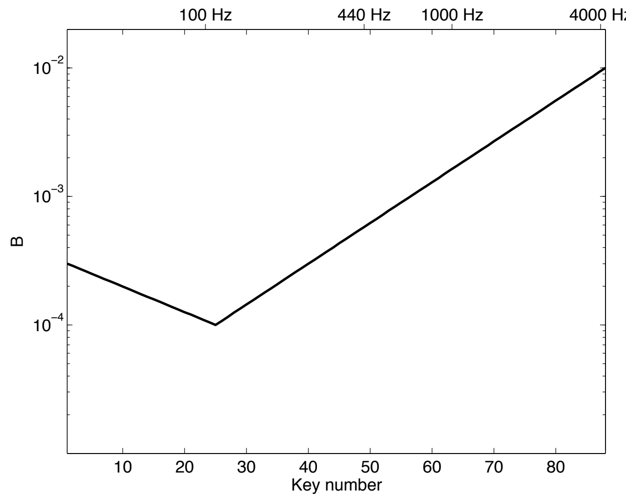


Figure 7: *The inharmonicity coefficient* $B$ *values used in the example piano synthesis model.*

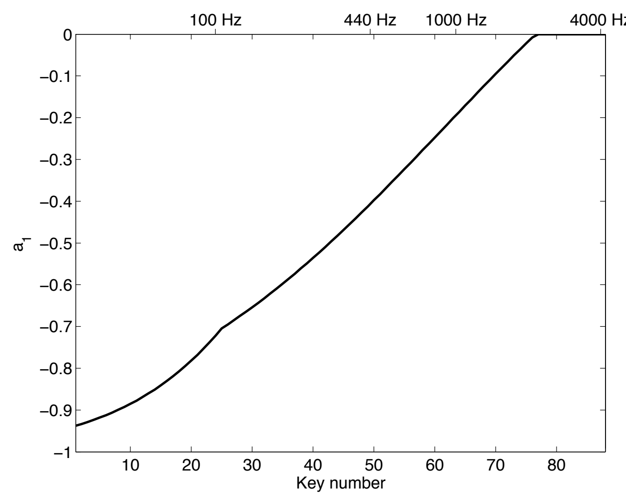


Figure 8: *The dispersion filter parameter* $a_1$ *values for the piano model when a cascade of* $M = 8$ *first-order allpass filters is used.*

in Figure 6, and the input signals were extracted from real piano samples [17]. The defined inharmonicity coefficient values used in the model are shown in Figure 7. In Figure 8, the corresponding dispersion filter coefficient values are presented. Moreover, the determined delay line length and the delay produced by the dispersion filter are illustrated in Figure 9.

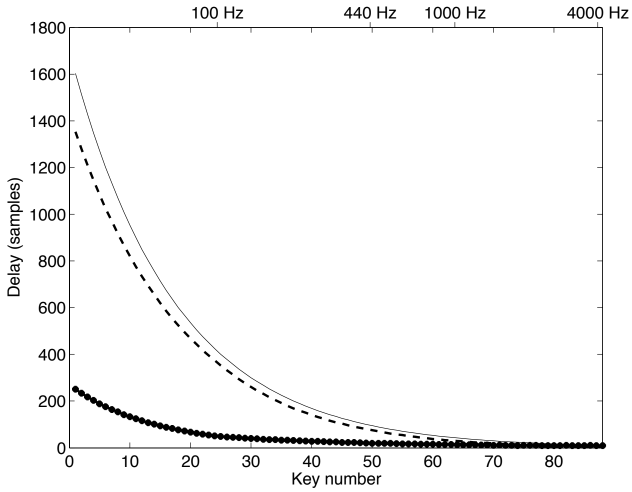


Figure 9: *The delay parameters for the piano model: the length of the delay line (dashed line), the length without the dispersion filter (solid line) and the difference between these two which is the delay produced by the dispersion filter (solid line with dots).*

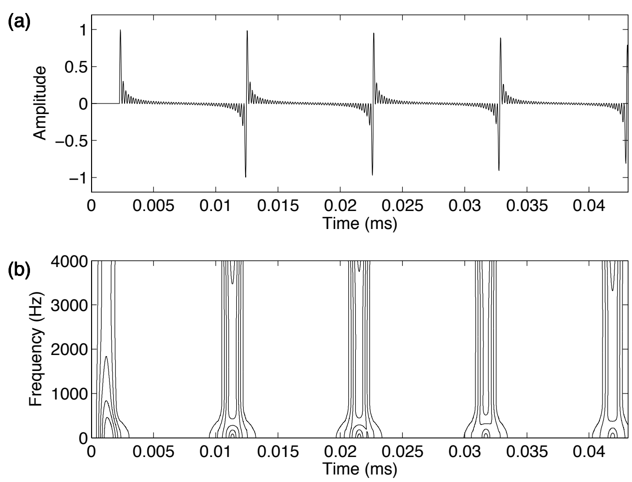


Figure 10: *(a) The waveform and (b) the time-frequency plot of a harmonic tone ($f_0 = 98.0$ Hz, key $\mathrm{G}_2$) produced with a string model with no dispersion filter. The excitation signal is simplified in order to emphasize the effect.*

When equation 3 resulted in a $D$ value less than 1 (for instance, key numbers 75-88 in Figure 9), $D$ was set to be 1, which corresponds to replacing the allpass filters with the transfer function $A(z) = 1$. The tuning filter was implemented with a first-order Thiran allpass filter, and a delay of one sample was moved from the delay line to the tuning filter in order to have the fractional delay parameter in the range from 1 to 2. It can be noticed that the tunable dispersion filter reduces the need for a delay line memory up of to 200 samples in this case (see Figure 9).

Figure 10 and Figure 11 illustrate how the dispersion filter affects the waveform and the spectral properties of the produced tone in time domain. This way of visualizing the effect of dispersion by using a short time window was suggested by Woodhouse [18]. The

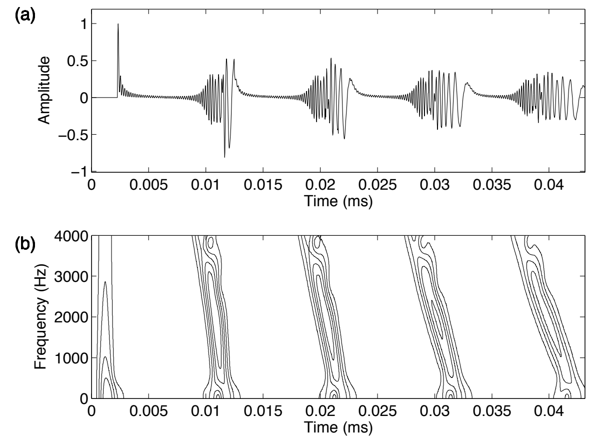


Figure 11: *(a) The waveform and (b) the time-frequency plot of an inharmonic tone ($f_0 = 98.0$ Hz, key $\mathrm{G}_2$, $B$ = 0.0001) produced with a string model with eight first-order allpass filters in cascade. The excitation signal is simplified in order to emphasize the effect.*

excitation of the synthetic tones contains 50 partials with sine starting phases produced by using additive synthesis. Two effects can be seen: the waveform of the tone becomes more spread out and the high frequencies travel faster in than the low frequencies in the feedback loop of the string model. Sound examples as well as Matlab code to calculate the dispersion filter coefficient are available at http://www.acoustics.hut.fi/demos/ext-disp/.

## 6. CONCLUSIONS

In this paper, a closed-form formula is proposed to be used in designing first-order allpass filters for dispersion modeling. The tunable dispersion filter design method that we introduced recently provides a technique which is extended in this work to determine the closed-form formula for an arbitrary number of first-order filters in cascade. It enables an easy design process for first-order dispersion filters, and, also, it provides real-time control over the dispersion phenomenon.

## 7. ACKNOWLEDGMENTS

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