THE FEATHERED CLARINET REED

Tamara Smyth, Jonathan Abel, Julius O. Smith III

CCRMA, Stanford University
Stanford, CA 94305, USA
{tamara|jonathan|jos}@ccrma.stanford.edu

ABSTRACT

In this research, a method previously applied to improve a digital simulation of the avian syrinx is adapted to the geometry of the clarinet reed. The clarinet model is studied with particular attention to the case when the reed beats again the lay of the mouthpiece, closing off air flow to the bore once each period. In place of the standard reed table which gives steady-state volume flow as a function of constant pressure difference across the reed, a more realistic dynamic volume flow model is proposed. The differential equation governing volume flow dynamics is seen to have a singularity at the point of reed closure, where both the volume flow and reed channel area become zero. The feathered clarinet reed refers to the method, first used in the syrinx, to smooth or feather the volume flow cutoff in a closing valve. The feathered valve eliminates the singularity and reduces artifacts in the simulated clarinet output.

1. INTRODUCTION

Many sounds are produced by coupling the mechanical vibrations of a source to the resonance of an acoustic tube. In the bird's vocal organ, the syrinx, air pressure from the lungs controls the oscillation of a membrane (by changing the pressure across the membrane), which creates a variable constriction through which air flows before reaching the upper bronchus and trachea [1, 2, 3]. Similarly, blowing into the mouthpiece of a clarinet will cause the reed to vibrate, narrowing and widening the airflow aperture to the bore. Sound sources of this kind are referred to as pressure-controlled valves and they have been simulated in various ways to create musical synthesis models of woodwind and brass instruments as well as animal vocal systems.

When airflow sets a valve into motion, it causes a change in the height of the valve channel and, if the motion is extreme, it can potentially close off the channel completely, creating a sudden termination in airflow. Examples of this occur when a clarinet reed beats against the lay of the mouthpiece or when the syrinx membrane touches the opposite wall of the upper bronchus. Such abrupt changes are difficult to synthesize because of undesirable time quantization effects that result when using relatively low audio sampling rates.

As described in [3] and [4], if the vibrating membrane in the syrinx were to reach to opposite wall of the bronchus, the membrane's flexible biological material would likely cause it to touch gradually, starting with the center bulge and the remainder settling gently on either side before finally closing off the channel. That is, as illustrated in Figure 1, instead of the channel being sealed the moment the membrane touches the opposite wall, it is more likely that flow will be able to seep through side corners and any other potential openings before the channel is closed completely.

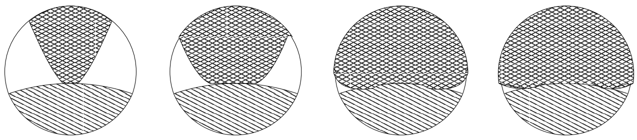

Figure 1: *A hypothetical view of a flexible biological membrane beating on the cartilage of the opposite valve wall. As the valve closes, it likely starts with the center bulge and the remainder gently settles on either side before the channel is closed off completely.*

In previous research [3], the behaviour of the differential equation governing volume flow through a syrinx valve was re-examined, paying particular attention to this troublesome transition between an open and closed valve. A closed-form solution for the time evolution of volume flow was given and used to derive an update for the volume flow which, in effect, sampled the continuous output of the differential equation governing volume flow. The result was a sort of leaky valve, with the leakage decreasing as the volume flow decreases, which smoothed or *feathered* the transition between the two states, significantly reducing the aliasing associated with the closing of the valve.

In this paper, also described in [4], improvements made to the syrinx valve [3] are applied to the clarinet reed, effectively feathering the beating clarinet reed. Though the cane reed is much more rigid than the bird's syrinx membrane, its simulation can benefit from the same principles of feathering—particularly when meeting the demands of relatively low audio sampling rates.

We begin by describing the current status of the clarinet simulation which employ a so-called *reed table* to relate the pressure difference across the valve to the volume flow through the valve channel. This quasi-static model ignores the dynamic relationship between the pressure difference and volume flow, yielding a set volume flow for each value of input pressure difference. We address this shortcoming by then introducing a dynamic model for airflow through the reed channel, and replace the reed table with differential equations governing displacement of the reed and the resulting volume flow through the valve channel. With the dynamic model in place, the feathered valve can then be incorporated, taking into account the geometry of the clarinet valve.

2. THE QUASI-STATIC CLARINET MODEL

Most current models of the clarinet reed are implemented using a lookup table which matches values for flow with the pressure

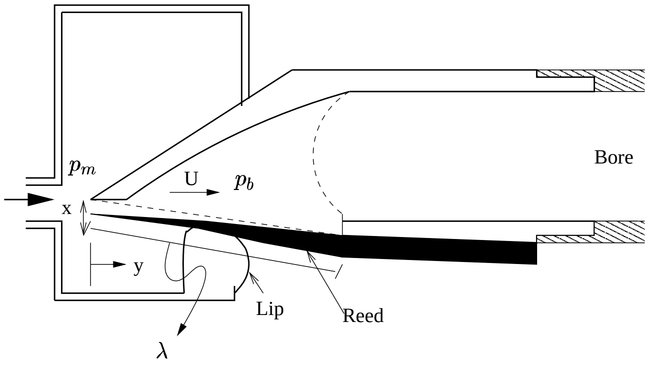

Figure 2: *A simplified diagram of a clarinet reed. The variable p_m represents the mouth pressure, p_b is the pressure in the bore, U is the volume flow, x is the displacement of the reed, y indicates the position along the reed and λ is the length of the unclamped end of the reed.*

drop across the reed valve. This is known as a quasi-static model since the value of flow, U, is established by using a lookup table (see Figure 3) relating pressure difference and volume flow under constant-flow conditions [5]. One of the benefits of the reed table approach is that it produces very satisfying results with low computational cost. One difficulty however, occurs when the reed beats against the lay of the mouthpiece: since the point of collisions between the reed and the lay is often too abrupt, the sound produced can be metallic and artificial. Furthermore, in the case that the reed beats against the lay, terminating air flow every cycle, it is clear that a static model is not entirely accurate [6].

An excellent description of the quasi-static clarinet reed model was published by Dalmont, Gilbert and Ollivier in [7], by Fletcher and Rossing in [8], and by Hirschberg et al. in [9], and is summarized here to give context to the discussion that follows.

The steady flow through a valve is determined based on an input (or blowing) pressure p_m and a resulting output (or mouthpiece) pressure p_b (see Figure 2). The difference between these two pressures is denoted Δp, and is related to volume flow via the stationary Bernoulli equation [8],

$$U = A\sqrt{\frac{2\Delta p}{\rho}}, \tag{1}$$

where A is the cross section area of the air column (and the jet) and ρ is the density of air. The steady state reed position, and therefore the jet cross-sectional area, is a function of the pressure difference alone, and (1) can be used to generate the reed table shown in Figure 3.

The geometry of the clarinet valve is given by the width of the reed channel w and the height of the opening H (or alternatively, the distance between the reed and the lay). The area of the valve opening A is therefore given by

$$A = wH. \tag{2}$$

The motion of the reed follows the familiar equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_r^2(x - x_0) = \frac{A}{m}\Delta p, \tag{3}$$

where γ is a damping coefficient, m is the effective mass of the reed and ω_r is the reed's resonant frequency. Since ω_r is related to

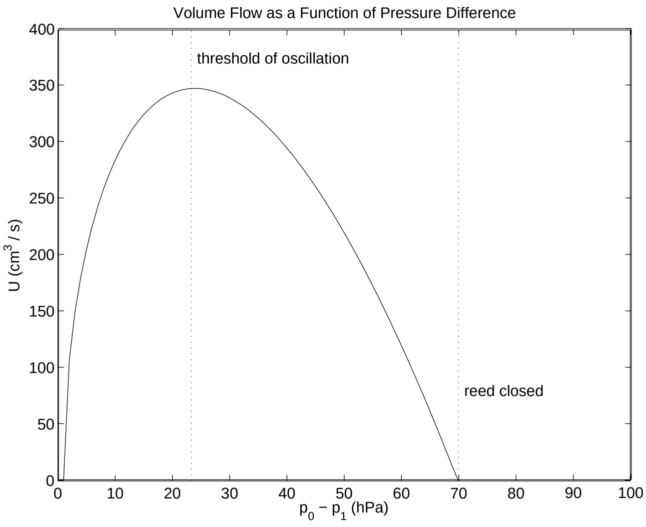

Figure 3: *The reed table provides a value for volume flow U corresponding to a change in pressure across the reed. The region of oscillation is between the two dotted lines.*

the stiffness and the mass by

$$\omega_r = \sqrt{\frac{\kappa}{m}}, \tag{4}$$

where κ is a constant describing the reed stiffness (in Pa/m), equation (3) can be rewritten for convenience as

$$\mu\frac{d^2x}{dt^2} + \mu g\frac{dx}{dt} + \kappa x = \Delta p, \tag{5}$$

where μ is mass per meter square and g is a viscous-damping coefficient (in s^{-1}) [7]. In the quasi-static model, the time derivatives in (5) are set to zero, rendering the mechanical reed effectively massless, with the stiffness being the only reactive element. The equation for the quasi-static reed therefore becomes

$$x = \frac{\Delta p}{\kappa}. \tag{6}$$

If H_0 is the equilibrium opening, that is, the opening of the valve in the absence of flow, the displacement of the reed determines the valve opening, H, by

$$H = H_0 - x. \tag{7}$$

From (6) and (7), the steady-state pressure difference corresponding to a just closed reed is determined by setting the opening to zero, that is setting the displacement to its maximum, $x = H_0$. Stated mathematically,

$$H_0 - H \quad = \quad x = \frac{\Delta p}{\kappa}, \tag{8}$$

$$\Delta p_{max} \quad = \quad \kappa H_0, \qquad \text{when } H = 0, \tag{9}$$

where Δp_{max} represents the maximum pressure difference, below which the valve is open. By applying (7) and (8), the area of the reed opening becomes

$$A = w\left(H_0 - \frac{\Delta p}{k}\right) \tag{10}$$

Quantity	Symbol	Value Range
Equilibrium reed opening	H_0	$0.04 - 0.1$ cm
Reed stiffness	κ	$800 - 1300$ hPa/cm
Effective width of jet	αw	$1.2 - 1.8$ cm
Maximum volume flow	U_{max}	$200 - 600$ cm^3/s
Flow on the reed	ν	0.9 cm
Damping coefficients	γ	1000/s
Density of air	ρ	0.0000012 kg/cm^3
Frequency of reed	f_1	1045 Hz
Reed length	λ	3.4 cm

Table 1: *Example value ranges for variables of the quasi-static clarinet model (some values are taken from [7]).*

which can be further reduced to

$$A = wH_0 \left(1 - \frac{\Delta p}{\Delta p_{max}}\right) \tag{11}$$

by applying (9). The stationary volume flow from (1) therefore becomes

$$U = w_0 \left(1 - \frac{\Delta p}{\Delta p_{max}}\right) \sqrt{\frac{2\Delta p}{\rho}}. \tag{12}$$

Note that a pressure difference of $\Delta p_{max}/3$ gives the maximum value for steady state flow U_{max},

$$U_{max} = \frac{2}{3} w H_0 \sqrt{\frac{2\Delta p_{max}}{3\rho}}. \tag{13}$$

It is just above this value of differential pressure that the reed can oscillate in response to an applied pressure [7].

If the pressure difference is greater than P_{max}, it is assumed that the reed is closed, and there is no flow through the valve channel (see Figure 3) and U is set to zero. This handling of the flow between open and closed valves can be improved by feathering the collision between the open and closed states.

3. REPLACING THE REED TABLE WITH THE DYNAMIC MODEL

Since volume flow doesn't instantly respond to changes in differential pressure, a dynamic model is needed. Before applying the *feathered* beating reed, the equations for flow and displacement in the quasi-static model must first be replaced with their corresponding differential equations, incorporating the appropriate valve geometry for the clarinet.

3.1. Volume Flow

The strategy for determining the volume flow derivative in the clarinet reed model is similar to that for the syrinx discussed in [3]. In the clarinet reed, only a short section of length ν along the y axis (see Figure 2) is in contact with the flow before the flow separates from the surface of the reed and forms a jet. The force on a thin slice dy along this part of the reed is given by

$$F = A(y;x)\Delta p(y). \tag{14}$$

where $A(y;x)$ is the area of the valve channel at this position and $\Delta p(y)$ is the pressure drop across this section of the reed.

This force is applied to a mass of

$$m = \rho A(y;x)dy, \tag{15}$$

where ρ is the air density and $A(y;x)dy$ is the volume to which the force is applied. Newton's second law, $F = ma$, can then be applied to (14) and (15) to obtain

$$A(y;x)\Delta p(y) = \rho A(y;x) dy \frac{dv}{dt}, \tag{16}$$

where acceleration is given by the time derivative of the particle velocity, dv/dt, assumed constant over this section dy of the reed.

Since volume flow is equal to particle velocity scaled by area, the expression for differential pressure as a function of position y along the reed channel is given by

$$\Delta p(y) = \rho \frac{dU}{dt} dy / A(y;x). \tag{17}$$

Equation (17) is then integrated over the length of the channel to obtain

$$p(0) - p(\nu) = \rho \frac{dU}{dt} \int_{y=0}^{y=\nu} dy / A(y;x), \tag{18}$$

where $y = 0$ is the channel entrance and $y = \nu$ is the point at which the flow separates from the surface of the reed and forms a jet.

The pressure at the channel entrance is obtained using Bernoulli's equation given by

$$p(y) = p_0 + \frac{\rho}{2}\left[v_0^2 - v(y)^2\right], \tag{19}$$

where v is the particle velocity. Again substituting volume flow divided by area for particle velocity, the pressure at the channel entrance $p(0)$ is given by

$$p(0) = p_m - \frac{\rho}{2}\left(\frac{U}{A(0;x)}\right)^2, \tag{20}$$

where p_m is the mouth pressure. Since the pressure at the point of flow separation $p(\nu)$ is equal to the bore pressure p_b, equation (18) becomes

$$p_m - p_b - \frac{\rho}{2}\left(\frac{U}{A(0;x)}\right)^2 = \rho \frac{dU}{dt} \int_{y=0}^{y=\nu} dy / A(y;x). \tag{21}$$

The differential equation governing volume flow is then given by

$$\frac{dU}{dt} = (p_m - p_b)\frac{A(x)}{\nu\rho} - \frac{U^2}{2\nu A(x)}, \tag{22}$$

where the flow is assumed to be in contact with the reed for a distance of ν, at an assumed constant area equal to

$$A(x) = A(0;x) = w(H_0 - x), \tag{23}$$

where w is the width of the reed (or jet) and H_0 is the opening of the valve channel in the absence of flow.

Note that in steady state, the dynamic equation for air flow (22) reduces to (1). Setting the time derivative dU/dt to zero, implies

$$(p_m - p_b)\frac{A(x)}{\nu\rho} = \frac{U^2}{2\nu A(x)}. \tag{24}$$

Solving for the steady state flow, we get

$$U = A(x)\left[\frac{2}{\rho}(p_m - p_b)\right]. \tag{25}$$

3.2. Reed Displacement

The equation for displacement is determined by considering the force acting on the clarinet reed. If the reed is rigid and hinged with spring constant k at a point far from the mouth side of the lip, let λ be the length of the reed, roughly along the y axis, which sees the mouth pressure p_m.

There is a force closing the reed which is given by

$$F_m = w\lambda p_m, \tag{26}$$

and in contrast, the force on the bore side of the reed away from the jet, given by

$$F_b = -w(\lambda - \nu)p_b, \tag{27}$$

where ν is defined as before, forces the reed open. The force applied by the flow (which also forces the reed open) is found by integrating the pressure along the flow and is given by

$$F_U = -w \int_{y=0}^{y=\nu} \left(p_m - \frac{\rho}{2} \left(\frac{U}{A(y;x)} \right)^2 \right) dy. \tag{28}$$

The overall force acting of the reed is obtained by summing (26), (27) and (28) and is given by

$$F = w(\lambda - \nu)(p_m - p_b) - w\nu \frac{\rho}{2} \left(\frac{U}{A(x)} \right)^2. \tag{29}$$

Once the force is known, the displacement of the reed is obtained using the familiar differential equation (comparable to 3)

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \frac{k}{m}(x - x_0) = \frac{F}{m}, \tag{30}$$

where F is defined by (29).

3.3. Model Aliasing

In simulating the valve, care must be taken in computing the volume flow between open and closed states. This is made more difficult by the singularity in the equation for the volume flow derivative (22) as the valve opening approaches zero.

Frequently this type of situation is handled by solving the equation only when it is stable,and substituting a fixed value at the point of singularity. It is well known, however, that switching based on a level threshold causes aliasing in discrete time signals. In this case, aliasing is caused by setting the volume flow $U(t)$ to zero when the valve closes.

As illustrated by the magnified plot of volume flow in Figure 4, $U(t)$ (the dashed line) is forced to zero on a sample boundary. Regardless of the value of $U(t)$ predicted by updating (22), air volume flow is still being set to zero when the valve is closed.

Theoretically, this approach seems correct, since no air should flow through a closed valve. Truncating the volume flow on sample boundaries, however, is problematic (both in the static and dynamic model). Depending on the period of the signal, the clipping may not happen at the correct phase and aliased components will be generated. This is illustrated in Figure 5 which shows a sinusoid and its truncated version along with their respective power spectra. Aliased components appear as peaks at nonharmonic frequencies.

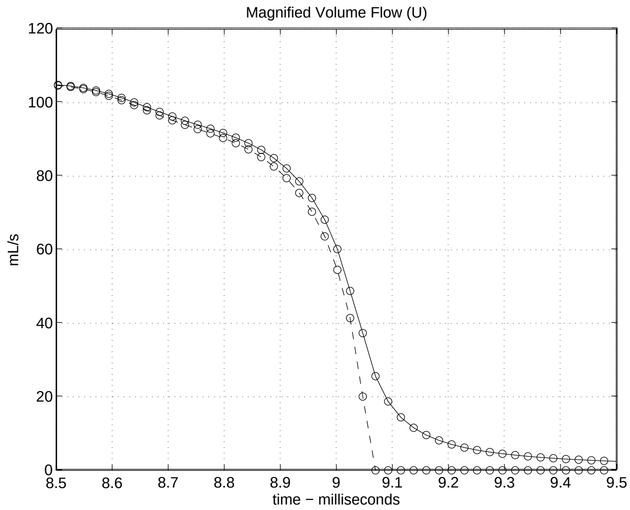

Figure 4: *A magnified view of volume flow showing truncation on a sample boundary (dashed line) and a more desirable gradual slope in the flow update (solid line) when the valve closes.*

4. IMPROVING THE DYNAMIC MODEL BY FEATHERING THE BEATING REED

The difficulty with discretizing (22) in the presence of small valve areas is illustrated in Figure 6. Since the slope of $U(t)$ is decreasing with decreasing volume flow (this can be seen in Figure 4), predictions of the slope based on (22) tend to overshoot zero volume flow [3]. It would therefore be preferable to use a small area solution of $dU(t)/dt$ to update the volume flow when the valve is closing. In [3] the small area solution to the volume flow update is solved for the syrinx geometry; here it is solved for the clarinet reed.

When the valve channel area $A(t)$ is sufficiently small, the first term of (22) can be ignored and the differential equation for $U(t)$ is approximated by

$$\frac{dU}{dt} \approx -\frac{U^2}{2\nu A(t)}, \qquad A(t) \ll 1, \tag{31}$$

which is in the form of a so-called *Bernoulli differential equation* [10]. Though this differential equation is nonlinear in $U(t)$, it may be converted to a linear form by the substitution

$$W(t) = \frac{1}{U(t)}. \tag{32}$$

Writing (31) in terms of $W(t)$ gives the following new differential equation for $U(t)$

$$\frac{dU}{dt} = -\frac{1}{W(t)^2} \frac{dW}{dt}, \tag{33}$$

where

$$\frac{dW}{dt} = \frac{1}{2\nu A(t)}; \tag{34}$$

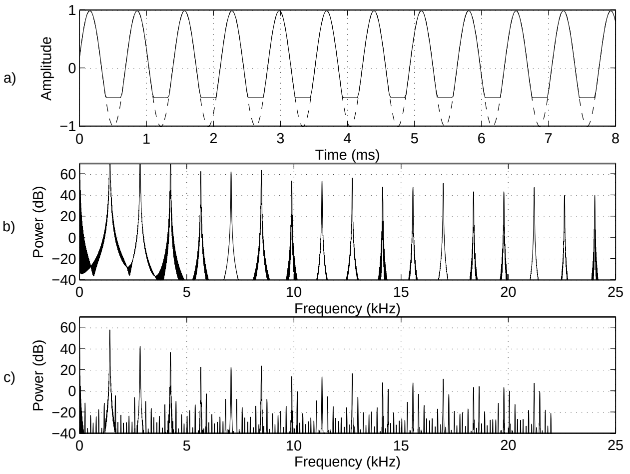

Figure 5: *Figure (a) shows a full and truncated version of a sine wave. Figure (b) shows the desired power spectrum of the truncated waveform, and Figure (c) shows the artifacts in the spectrum if the truncation is done too abruptly for the sampling rate used.*

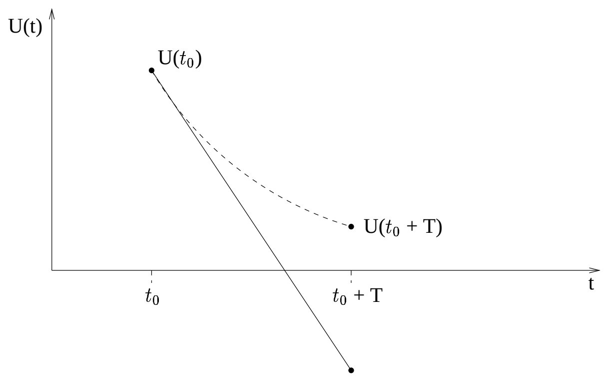

Figure 6: *In the case of a large sampling period T, updating the volume flow using (22) can cause U to overshoot. The dotted line represents the actual value of U.*

This equation is easily integrated to solve for volume flow:

$$W(t) = \int^t \frac{d\tau}{2\nu A(\tau)} + C, \tag{35}$$

$$U(t) = \frac{1}{\displaystyle\int^t \frac{d\tau}{2\nu A(\tau)} + C}, \tag{36}$$

where the constant of integration C may be set given knowledge of $U(t)$ at a particular time t_0 and

$$U(t) = \frac{U(t_0)}{1 + U(t_0)\displaystyle\int_{t_0}^t \frac{d\tau}{2\nu A(\tau)}}. \tag{37}$$

Note that when the area $A(t)$ is small, the integral in the denominator of (37) is large, and any initial positive value of volume flow is quickly reduced to zero without crossing zero, as would be expected for a closing valve. This observation provides justification for having zero volume flow when the valve area is zero. The small valve area solution to (37) suggests a possible alternative to truncating U when the valve is closed (which would be otherwise necessary given the singularity in (22)). If the valve were slightly leaky, e.g.,

$$\tilde{A}(t) = A(t) + \lambda, \tag{38}$$

for a small leakage area λ, the singularity at zero area would be avoided, and the volume flow behaviour would be relatively unchanged. However, it is not sufficient to use a leaky valve in place of one that is truncated because though this may reduce the slope of $U(t)$ it also introduces the undesirable behaviour of volume flow oscillating about zero.

In order to see how this solution should be incorporated into the volume flow update, consider the value of $U(t)$ at time $t_0 + T$, where T is the sampling period. Given the small area solution for volume flow (37), but in a more convenient form

$$U(t) = \left[\frac{1}{U(t_0)} + \int_{t_0}^t \frac{1}{2\nu A(\tau)} d\tau\right]^{-1}, \tag{39}$$

the valve channel area can be substituted by $A(t_0)$, since it is assumed to be constant during the time interval $[t_0, t_0 + T]$. Substituting into (39) we obtain

$$U(t) = \left[\frac{1}{U(t_0)} + \frac{1}{2\nu A(t_0)}(t - t_0)\right]^{-1}, \tag{40}$$

and the volume flow at $t_0 + T$ is

$$U(t_0 + T) = U(t_0)\left[1 + \frac{U(t_0)}{2\nu A(t_0)}T\right]^{-1}. \tag{41}$$

Using the first order backwards difference approximation, the new differential equation for $U(t)$ becomes

$$\frac{dU}{dt} = \frac{U(t_0 + T) - U(t_0)}{T} \tag{42}$$

$$= -\frac{U(t_0)^2}{2\nu A(t_0)} \cdot \left[1 + \frac{U(t_0)}{2\nu A(t_0)}T\right]^{-1}. \tag{43}$$

Comparing the form of (43) to (31) note that the Bernoulli terms are identical, save a factor of $[1 + U(t_0)/2\nu A(t_0)T]^{-1}$. This factor has the effect of reducing the derivative in the presence of small channel areas or large sample periods. Rewriting (43) gives

$$\frac{dU}{dt} = -\frac{U(t_0)^2}{2\nu A(t_0) + U(t_0)T}. \tag{44}$$

Note that in this form the Bernoulli term is similar to that of (31), with a valve having increased area. In other words, it has become a leaky valve whose leakage increases with increasing volume flow. The final step is to replace the second term in (22) with this new Bernoulli term (44) to obtain

$$\frac{dU}{dt} = (p_m - p_b)\frac{A(t_0)}{\nu\rho} - \frac{U(t_0)^2}{2\nu A(t_0) + U(t_0)T}. \tag{45}$$

which is the final feathered differential equation for volume flow.

5. CONCLUSIONS

The differential equation (22) describing the behaviour of volume flow (22) can be numerically unstable because of the singularity in the Bernoulli term when the valve closes. Since abruptly setting the flow to zero causes aliasing, the problem is addressed by incorporating the new small-area solution for $U(t)$. The volume flow is now updated in a way which produces smoother transitions between open and close valves.

By feathering the collisions of the beating reed in the clarinet simulation, the sound is greatly improved. This is illustrated in the output spectrum of the static model (Figure 8) and that of the feathered dynamic model (Figure 7) where the aliasing is significantly reduced.

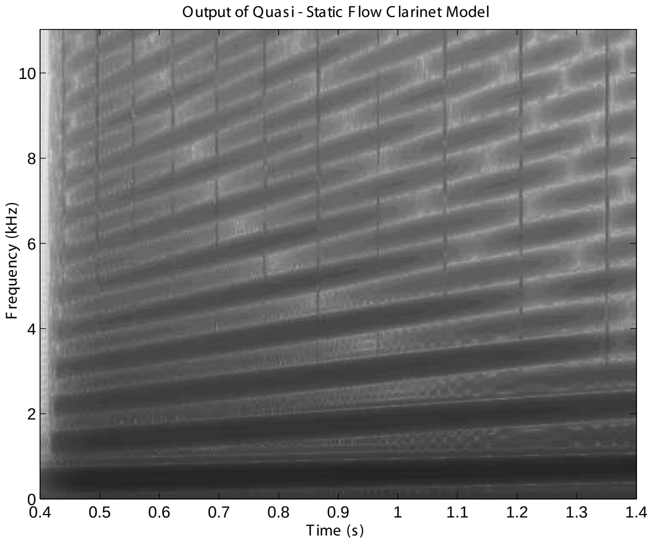

Figure 7: *Output of the quasi-static clarinet model using a sampling rate of 44.1 kHz. Control parameter values were mouth pressure, 70 - 10 hPa, and frequency, 300-600 Hz. Lines through the spectrum illustrate undesirable artifacts.*

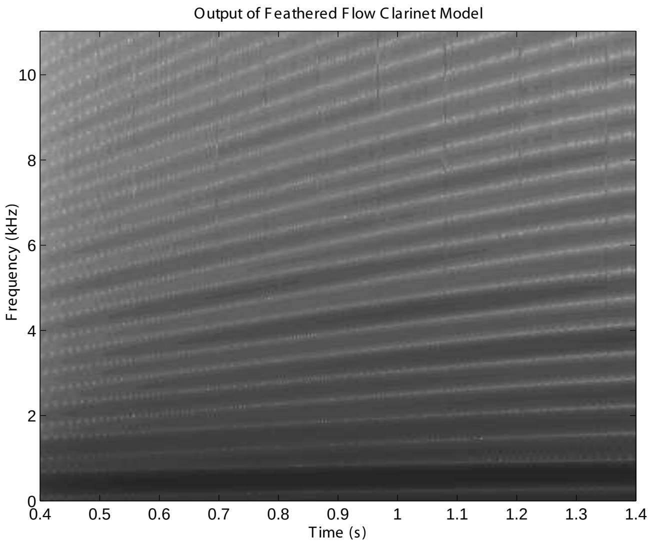

Figure 8: *Output of the "feathered" dynamic clarinet model with the same control parameters and the same sampling rate as Figure 7. The output is improved overall and almost free of artifacts.*

6. REFERENCES

[1] Tamara Smyth and Julius O. Smith, "The sounds of the avian syrinx—are they really flute-like?," in *DAFX 2002 Proceedings*, Hamburg, Germany, September 2002, International Conference on Digital Audio Effects.

[2] Tamara Smyth and Julius O. Smith, "The syrinx: Nature's hybrid wind instrument," in *CD-ROM Paper Collection*, Cancun, Mexico, September 2002, Pan-America/Iberian Meeting on Acoustics.

[3] Tamara Smyth, Jonathan Abel, and Julius O. Smith, "Discrete-time simulation of air-flow cut-off in pressure-controlled valves," New Paltz, New York, October 2003, IEEE Workshop on Applications of Signal Processing to Audio and Acoustics.

[4] Tamara Smyth, *Applications of Bioacoustics to Musical Instrument Technology*, Ph.D. thesis, Stanford University, April 2004.

[5] Julius O. Smith, *Digital Waveguide Modeling of Musical Instruments*, `ccrma.stanford.edu/~jos/waveguide/`, 2003.

[6] Maarten van Walstijn, *Discrete-Time Modelling of Brass and Reed Woodwind Instruments with Application to Musical Sound Synthesis*, Ph.D. thesis, University of Edinburgh, 2002.

[7] Jean-Pierre Dalmont, Joël Gilbert, and Sébastien Oliver, "Nonlinear characteristics of single-reed instruments: Quasistatic vlume flow and reed opening measurements," *Journal of the Acoustical Society of America*, vol. 114, no. 4, pp. 2253–2262, October 2003.

[8] Neville H. Fletcher and Thomas D. Rossing, *The Physics of Musical Instruments*, Springer-Verlag, 1995.

[9] A. Hirschberg, R. W. A. van de Laar, J. P. Marrou-Maurières, A. P. J. Wijnands, H. J. Dane, S. G. Kruijswijk, and A. J. M. Houtsa, "A quasi-stationary model of air flow in the reed channel of single-reed woodwind instruments," *Acustica*, vol. 70, no. 2, pp. 146–154, February 1990.

[10] Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, Inc, 1978.